MATHEMATICS OF OPERATIONS RESEARCH
Vol. 14, No. 1, February 1989
Printed in U.S.A.

PROBABILISTIC ANALYSIS OF THE MULTIDIMENSIONAL KNAPSACK PROBLEM*

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We analyse the multi-constraint zero-one knapsack problem, under the assumption that all coefficients are drawn from a uniform [0, 1] distribution and there are $m = O(1)$ constraints as the number of variables grows. We show that results on the single-constraint problem can be extended to this situation. Chiefly, we generalise a result of Lueker on the duality gap, and a result of Goldberg/Marchetti-Spaccamela on exact solvability. In the latter case, our methods differ markedly from those for the single-constraint result.

1. Introduction. This paper is concerned with a probabilistic analysis of the multidimensional 0–1 knapsack problem IP:

$$\text{(1.1a)} \qquad \text{Maximise} \sum_{j=1}^{n} p_j x_j$$

subject to

$$\text{(1.1b)} \qquad \sum_{j=1}^{n} w_{ij} x_j \leqslant b_i, \qquad i = 1, 2, \ldots, m,$$

$$\text{(1.1c)} \qquad 0 \leqslant x_j \leqslant 1, \qquad j = 1, 2, \ldots, n,$$

$$\text{(1.1d)} \qquad x_j \text{ integer}, \qquad j = 1, 2, \ldots, n.$$

Here $p_1, \ldots, p_n, w_{11}, \ldots, w_{mn}, b_1, \ldots, b_m$ are all nonnegative real numbers.

In our random model we have

$$\text{(1.2)} \qquad p_1, \ldots, p_n, w_{11}, \ldots, w_{mn}$$

are drawn independently from the uniform [0, 1] distribution, and

$$\text{(1.3)} \qquad b_i = \beta_i n, \qquad i = 1, 2, \ldots, m,$$

where $0 < \beta_i < 1/2$ are fixed, for $i = 1, 2, \ldots, m$.

The reason for assuming $\beta_i < 1/2$ in (1.3) is that a standard inequality (see Lemma 3.1) implies that

$$\sum_{j=1}^{n} w_{ij} \leqslant (1 + o(1))n/2, \qquad i = 1, 2, \ldots, m, \quad \text{(a.s.)}$$

where an event E_n occurs *almost surely* (a.s.) if $\lim_{n \to \infty} \Pr(E_n) = 1$.[1]

*Received April 12, 1985; revised October 5, 1987.
AMS 1980 subject classification. Primary: 90C10.
IAOR 1973 subject classification. Main: Programming: Integer.
OR/MS Index 1978 subject classification. Primary: 627 Programming/Integer.
Key words. Multidimensional knapsack problem, probabilistic analysis, branch-and-bound.
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[1] This is not the standard definition in probability theory, but is common in the probabilistic analysis of algorithms.

0364-765X/89/1401/0162$01.25
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When $m = 1$ we have the 0-1 knapsack problem. This has been extensively studied from a variety of viewpoints. Although it is an NP-Hard problem, there is a lot of empirical evidence suggesting that randomly generated 0-1 knapsack problems are usually easy to solve—see for example Balas and Zemel [1].

Recent theoretical results of Lueker [8] and Goldberg and Marchetti-Spaccamela [5] lend support to this.

There is empirical evidence that multi-dimensional knapsack problems with few constraints are also easy to solve—see for example [7]. The main aim of this paper is to extend the results of [5] and [8] to the case $m > 1$ but *fixed* and $n \to \infty$. The difficulty in the generalization lies in the relative complexity of linear programming with one constraint as against many, and of the geometry of two dimensions as against geometry of an arbitrary number of dimensions. This necessitates somewhat different methods from those of [5]. However, we believe that our methods are generally more natural than those of [5], although we do not reproduce all of the results claimed there.

Let v_{IP} denote the value of the optimal solution to IP.

We need to consider the linear programming relaxation $\mathrm{LP} = \mathrm{LP}(\mathbf{b})$, $\mathbf{b} = (b_1, b_2, \ldots, b_m)$ obtained by ignoring (1.1d). We let $v_{\mathrm{LP}} = v_{\mathrm{LP}}(\mathbf{b})$ denote the value of an optimal solution to this problem. (We write $\mathrm{LP}(\mathbf{b})$ as we will also need to consider linear programs with small changes to the b_i's.)

Our first result generalizes that of Lueker [8].

THEOREM 1.1.

$$\text{(1.4)} \qquad E(v_{\mathrm{LP}} - v_{\mathrm{IP}}) \leqslant a_1 \log^2 n/n,$$

$$\text{(1.5)} \qquad \Pr\left(v_{\mathrm{LP}} - v_{\mathrm{IP}} \geqslant t a_2 \log^2 n/n\right) \leqslant (1 - a_3)^t,$$

for some 'constants' $a_1, a_2 > 0$, $0 < a_3 < 1$ depending only on m and $\beta = \min(\beta_1, \ldots, \beta_m)$ and positive integers $t \leqslant -3\log n/\log(1 - a_3)$.

Our second result generalizes that of Goldberg and Marchetti-Spaccamela [5]:

THEOREM 1.2. *Given $\epsilon > 0$ there is an algorithm A_ϵ which runs in time $O(n^{d(\epsilon, m, \beta)})$ and with probability at least $1 - \epsilon$ solves IP exactly for some function $d(\epsilon, m, \beta) > 0$.*

We are also able to show, as in [5], that

THEOREM 1.3. *Let MAXCH denote the maximum number of variables that have different values in a pair of optimal solutions to IP and LP respectively.*

Then $\Pr(MAXCH \geqslant t\log n) \leqslant (1 - a_4)^t$ for some $a_4(m, \beta) > 0$, provided

$$t \log\left(\frac{t}{4e}\right) \geqslant 2(m + 1) \quad \text{and} \quad t \leqslant -\frac{6a_2 \log n}{\log(1 - a_3)}.$$

The "constants" a_1, a_2, a_3, a_4 that we construct vary rapidly with m.

2. Preliminaries. In this section we introduce some notation and make some preliminary observations.

Let $C = [0, 1]^{m+1}$ be the unit hypercube in R^{m+1}. Each variable x_j is associated with the point $X_j = (w_{1j}, w_{2j}, \ldots, w_{mj}, p_j) \in C$ and the points $X = \{X_1, \ldots, X_n\}$ are drawn independently from the uniform distribution in C. We let the slack variables for

(1.1b) be denoted by $x_{n+1}, \ldots, x_{n+m}$. Then X_{n+i} is the ith column of the $(m+1) \times (m+1)$ identity matrix.

Using the continuity of the distribution of X (which relegates equalities to null events) and the basic theory of linear programming with simple upper bounds (see for example Chvatal [2], in particular Chapter 8), the optimal solution $\mathbf{x}^0$ to LP(b) can be assumed to have the following form: with probability 1 there is a unique set $S \subseteq Z_{n+m} = \{1, 2, \ldots, n+m\}$, $|S| = m$, of *basic indices* and dual variables $u_1^0, u_2^0, \ldots, u_m^0 \geq 0$ such that

(2.1a) $$\bar{p}_j = p_j - \sum_{i=1}^{m} u_i^0 w_{ij} = 0, \qquad j \in S,$$

($\bar{p}_j$ is the reduced cost of x_j)

(2.1b) $$\bar{p}_j > 0 \to x_j^0 = 1,$$

(2.1c) $$\bar{p}_j < 0 \to x_j^0 = 0.$$

The values of x_j^0, $j \in S$ are then uniquely determined by (1.1b) with equality, as slacks have been introduced.

Consider a point $\mathbf{w} = (w_1, \ldots, w_m, p) \in R^{m+1}$ and let $\Pi = \Pi(\mathbf{b}) = (-u_1^0, \ldots, -u_m^0, 1)$. Then (2.1) can be rephrased as

(2.2a) $$\Pi \cdot \mathbf{w} = 0$$

is the equation of the unique hyperplane $H(\mathbf{b})$ in R^{m+1} determined by the origin and X_j: $j \in S$.

(2.2b) If X_j is 'above' $H(\mathbf{b})$ then $x_j^0 = 1$.

(2.2c) If X_j is 'below' $H(\mathbf{b})$ then $x_j^0 = 0$.

LEMMA 2.1. *Let Π, S be as above and $C_1 = C \cap \{\mathbf{w}: \Pi \cdot \mathbf{w} < 0\}$. Let $N = \{j: X_j \in C_1\}$ in the optimal solution to* (1.1). *Then, conditional on the (random) sets S, N and on the $X_l (l \in S)$, the X_j $(j \in N)$ are uniformly and independently distributed in C_1.*

PROOF. Let $\xi = (X_1, X_2, \ldots, X_n)$ be the totality of coefficients to (1.1) excluding $b_1, \ldots, b_m$. Then clearly ξ is drawn from a uniform distribution on the unit hypercube C^n in $R^{(m+1)n}$.

Now consider all $\binom{n+m}{m}$ possible choices for S. For any such S, let $I = Z_n - S$ and consider all the $2^{|I|}$ assignments of 0 (lower bound) or 1 (upper bound) to the x_j $(j \in I)$. The collection of such assignments over all S define all possible different (extended) bases to (1.1). There are clearly

$$r = \sum_{j=0}^{m} \binom{n}{j} \binom{m}{m-j} 2^{n-j}$$

such assignments in total.

Now these r extended bases partition C^n into r regions (some possibly empty) such that within a given region R, the corresponding basis is feasible and optimal. Note that, conditional on $\xi \in R$, ξ is uniform on R. Now R is completely determined by a set of (nonlinear) inequalities on the elements of ξ, corresponding to the linear programming feasibility and optimality conditions for the basis. These conditions depend only on S, N. Thus, conditional on S and N, $\xi \in R$ for some particular region R. Let u_i be the dual variables corresponding to the basis for R, and note that these are completely determined by the X_l $(l \in S)$ through (2.1a). If x_t is any nonbasic variable assigned 0 in R (i.e. $t \in N$), it follows that the only defining condition for R involving X_t is the optimality condition (2.1c), i.e. $X_t \in C_1$. (Note that the feasibility conditions do not involve the $X_j \in C_1$.) Conditional on the u_i, and hence only on the X_l $(l \in S)$, this is a simple linear inequality on X_t.

Thus, conditional on $\xi \in R$ and on its particular components X_l $(l \in S)$, X_t is uniform on C_1 and independent of all other X_j. This condition is clearly identical to that of the Lemma. ∎

Finally in this section we note that the dual to LP(b) is

$$\text{DLP(b)}: \quad \text{minimise } \sum_{i=1}^{m} b_i u_i + \sum_{j=1}^{n} v_j$$

$$\text{subject to } \sum_{i=1}^{m} w_{ij} u_i + v_j \geq p_j, \qquad j = 1, 2, \ldots, n$$

$$u_i, v_j \geq 0, \qquad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n.$$

By inspection, we see that

$$v_j = \max\left(0, p_j - \sum_{i=1}^{m} w_{ij} u_i\right)$$

in an optimum solution, and so only $u_1, u_2, \ldots, u_m$ need be given for a solution to be defined.

3. Proof of Theorem 1.1.

This section is devoted to the proof of Theorem 1.1. Let us first define

(3.1a) $$\beta = \min\{\beta_1, \ldots, \beta_m\}, \quad \hat{\beta} = \max\{\beta_1, \ldots, \beta_m\} < 1/2,$$

(3.1b) $$a = 9(m+5)/\beta^m, \quad k = \lfloor a\beta^m \log n/4 \rfloor,$$

(3.1c) $$b_i' = b_i - k\beta/2, \qquad i = 1, 2, \ldots, m.$$

These definitions are placed here for ease of reference.

The following lemma is extremely useful in proving that sums of random variables "behave as expected". It is implicit in Theorem 1 of Hoeffding [6].

LEMMA 3.1. *Let $X_i \in [0,1]$, $i = 1, 2, \ldots, n$ be independent random variables. Let $\bar{X} = (X_1 + X_2 + \cdots + X_n)/n$ and let $\mu = E(\bar{X}) \leq \frac{1}{2}$. Then*

(3.2a) $$\Pr(\bar{X} \leq (1-\epsilon)\mu) \leq e^{-\frac{1}{2}\epsilon^2 n\mu}, \qquad 0 \leq \epsilon \leq 1,$$

(3.2b) $$\Pr(\bar{X} \geq (1+\epsilon)\mu) \leq e^{-\frac{1}{3}\epsilon^2 n\mu}, \qquad 0 \leq \epsilon \leq 1. \quad ∎$$

The proof of Theorem 1.1 goes along these lines: we solve LP(b′), b′ as in (3.1c). We round down this solution to obtain a solution to LP(b″) for some b″. We then try to select some variables T, currently at zero, which we can increase to one. As the following lemma shows, we need to find T so that (i) the variables in T have small reduced costs, (ii) their addition makes the binding constraints nearly tight.

LEMMA 3.2. *Let $x_1^*, \ldots, x_n^*$ be the optimal basic solution to LP($\mathbf{b}'$) and let $u_1^*, \ldots, u_m^*$ define an optimal solution to DLP($\mathbf{b}'$).*

*Let S be the optimal set of basic variables and let $N = \{j \in Z_n - S: x_j^* = 0\}$. Let N' be $Z_n - (N \cup S)$, which equals $\{j \in Z_n: x_j^* = 1\}$ with probability 1. Let $\bar{p}_j$ be as defined in (2.1a), but with 0 superscripts replaced by *'s.*

For $T \subseteq N$ let

$$\delta_i = b_i - \sum_{j \in N' \cup T} w_{ij}, \qquad i = 1, 2, \ldots, m.$$

Let $V_T = \sum_{j \in N' \cup T} p_j$ = the value of the solution to IP obtained by putting $x_j = 1$ for $j \in N' \cup T$, and zero otherwise.

Note that this is feasible if and only if $\delta_i \geqslant 0$ for $i = 1, 2, \ldots, m$. If $M^* = \{i: u_i^* > 0\}$ then

$$V_T \geqslant V_{LP}(\mathbf{b}) + \sum_{j \in T} \bar{p}_j - (1 + 1/\sqrt{n}) \sum_{i \in M^*} \delta_i/\beta. \tag{3.3}$$

PROOF. Note first that by basic theory of linear programming, if $b_i'' = \sum_{j \in N'} w_{ij}$, $i = 1, 2, \ldots, m$, then setting $x_j = 1$ for $j \in N'$ and $x_j = 0$ for $j \in V_n - N'$ solves LP($\mathbf{b}''$). Further, $u_1^*, \ldots, u_m^*$ defines an optimal solution to DLP($\mathbf{b}''$) also. Thus

$$\begin{aligned} V_T &= V_{\mathrm{LP}}(\mathbf{b}'') + \sum_{j \in T} p_j \\ &= V_{\mathrm{LP}}(\mathbf{b}'') + \sum_{j \in T} \left(p_j - \sum_{i=1}^m w_{ij} u_i^* \right) + \sum_{i=1}^m u_i^* \sum_{j \in T} w_{ij} \\ &= V_{\mathrm{LP}}(\mathbf{b}'') + \sum_{j \in T} \bar{p}_j + \sum_{i=1}^m u_i^* (b_i - b_i'' - \delta_i) \\ &\geqslant V_{\mathrm{LP}}(\mathbf{b}) + \sum_{j \in T} \bar{p}_j - \sum_{i=1}^m u_i^* \delta_i \end{aligned}$$

using the fact that $u_1^*, \ldots, u_m^*$ is a subgradient of V_{LP} at $\mathbf{b}''$.

Now

$$V_{\mathrm{LP}}(\mathbf{b}') = \sum_{i \in M^*} b_i' u_i^* + \sum_{j=1}^m v_j^* \leqslant n$$

and so for n large

$$\left(1 - \frac{1}{2\sqrt{n}}\right) \beta \sum_{i \in M^*} u_i^* \leqslant 1 \tag{3.4}$$

which is weakened to

$$u_i^* \leqslant \left(1 + \frac{1}{\sqrt{n}}\right) \Big/ \beta, \qquad i \in M^*,$$

to yield (3.3). ∎

We must now show that a suitable T can be found with high enough probability. For positive integer $t \leqslant L$, where $L = \lceil (\log n)^2 \rceil$,

$$S_t = C \cap \left\{ (w_1, \ldots, w_n, p): -p + \sum_{i=1}^m u_i^* w_i \in [(t-1)\delta, t\delta] \right\}$$

where $\delta = a \log n/n$ (see 3.1b) and $\mathbf{u}^*$ is as in Lemma 3.2.

The sets S_t form parallel 'slabs' of 'vertical' thickness δ lying underneath the hyperplane $H(\mathbf{b}')$.

LEMMA 3.3. (a) *For positive integer t, let E_t denote the following event: $\exists T$: $|T| = k$ (see 3.1b) such that*

$$j \in T \to X_j \in S_t, \tag{3.5a}$$

$$\sum_{j \in T} w_{ij} \in [c_i - \theta, c_i], \qquad i = 1, 2, \ldots, m, \quad \textit{where} \tag{3.5b}$$

$$\theta = \frac{(\beta - \alpha)}{\sqrt{6}} \left(\frac{(\pi k)^{(m+1)/2}}{4^k} \right)^{1/m}, \qquad \alpha = L\delta/(1 - 2\beta(1 + 1/n^{1/5})) \quad \textit{and}$$

$$\begin{aligned} c_i &= b_i - b_i'', & i &\in M^* \\ &= b_i - b_i', & i &\notin M^*. \end{aligned}$$

Then

$$\Pr\left(\bigcap_{i=1}^t \bar{E}_i \right) \leqslant \left(1 - \frac{1 + o(1)}{1 + (2/\sqrt{3})^m} \right)^t + O(1/n^4) \quad \textit{for } 1 \leqslant t \leqslant L.$$

(b) *If E_t occurs then, for large n,*

$$V_{LP} - V_{IP} \leqslant 21\beta^{-m}(m+5)^2 t (\log n)^2/n.$$

PROOF. (a) We first show that, for n large,

$$\Pr\left(\sum_{i=1}^m u_i^* \leqslant 1 - 2\beta(1 + 1/n^{1/5}) \right) = O(1/n^4). \tag{3.6}$$

Let $\sigma = \sum_{i=1}^m u_i^*$. If $\sigma < 1$ let $C^+ = C \cap \{(\mathbf{w}, p): p > \sigma\}$. Clearly $\mathrm{Vol}(C^+) = 1 - \sigma$ and $X_j \in C^+$ implies $p_j - \sum_{i=1}^m u_i^* w_{ij} > 0$. Hence

$$X_j \in C^+ \quad \text{implies } x_j^* = 1, \quad \text{if } \sigma < 1. \tag{3.7}$$

Let now $C_s^+ = C \cap \{(\mathbf{w}, p): p > s/\sqrt{n}\}$ for $s = 1, 2, \ldots, \sqrt{n} - 1$. We show that, for n large,

$$\Pr\left(\exists s, i: \sum_{X_j \in C_s^+} w_{ij} \leqslant (1 - \epsilon)(n - s\sqrt{n})/2 \right) = O(1/n^4) \tag{3.8}$$

where $\epsilon = 1/2n^{1/5}$.

Now since $C^+ \supseteq C_s^+$ for $s > \sigma\sqrt{n}$ we find that (3.6) follows from (3.7), (3.8).
To see this note that (3.7) and (3.8) imply

$$\beta_i n \geqslant \sum_{x_j \in C^+_{\lceil \sigma\sqrt{n}\rceil}} w_{ij} > (1-\epsilon)\left(n - \lceil \sigma\sqrt{n}\rceil\sqrt{n}\right)/2$$

and (3.6) follows for large n.

To prove (3.8) consider a fixed s, i. We define the random variables Y_j, $j = 1, 2, \ldots, n$ by

$$Y_j = w_{ij} \quad \text{if } X_j \in C_s^+,$$
$$= 0 \quad \text{otherwise.}$$

Now $E(Y_j) = (1 - s/\sqrt{n})/2$ and so applying Lemma 3.1

$$\Pr\left(\sum_{j=1}^n Y_j \leqslant (1-\epsilon)(n - s\sqrt{n})/2\right) \leqslant \exp\left(-(n - s\sqrt{n})\epsilon^2/4\right)$$

$$\leqslant e^{-\epsilon^2\sqrt{n}/4} = O(1/n^4).$$

Next let

$$B = \left\{(w_1, \ldots, w_m) \colon \alpha \leqslant w_i \leqslant \left(1 - 1/\sqrt{n}\right)\beta \text{ for } i = 1, 2, \ldots, m\right\}.$$

Let $\hat{S}_t = \{(w_1, \ldots, w_m, p) \in S_t \colon (w_1, \ldots, w_m) \in B\}$.

We note first that if $\mathbf{w} \in B$, (3.4) holds and $\sum_{i=1}^m u_i^* \geqslant 1 - 2\beta(1 + 1/n^{1/5})$ then

$$\text{(3.9a)} \qquad \left[\sum_{i=1}^m u_i^* w_i - t\delta,\ \sum_{i=1}^m u_i^* w_i - (t-1)\delta\right] \subseteq [0,1] \quad \text{for } 1 \leqslant t \leqslant L.$$

Thus each point of $\hat{S}_t$ is part of a vertical line segment of length δ contained entirely in $\hat{S}_t$. Thus

$$\text{(3.9b)} \qquad \operatorname{Vol}(\hat{S}_t) \geqslant (\beta' - \alpha)^m \delta, \quad \text{where } \beta' = \beta\left(1 - 1/\sqrt{n}\right).$$

We now use (3.9) to show that

$$\text{(3.10)} \quad \Pr\left(\exists\, 1 \leqslant t \leqslant L \text{ such that } \left|\{j \colon X_j \in S_t\}\right| \leqslant (\beta^m a \log n/2)\right) = O(1/n^4).$$

Unfortunately, because of conditioning problems, we cannot say immediately, $\Pr(X_j \in S_t) = \operatorname{Vol}(S_t)$ and apply Lemma 3.1. We are instead forced to prove (3.10) for all possible positions of the plane $H(\mathrm{b}')$.

Now for fixed $S \subseteq Z_{n+m}$ let $H(S) = \{(\mathbf{w}, p) \colon p = \sum_{i=1}^m u_i w_i\}$ denote the hyperplane through the origin and $\{X_j \colon j \in S\}$. Let $A(S)$ be the event

(i) $u_1, \ldots, u_m \geqslant 0$.
(ii) $(1 - 1/2\sqrt{n})\beta\sum_{i=1}^m u_i \leqslant 1$.
(iii) $\sum_{i=1}^m u_i \geqslant 1 - 2\beta(1 + 1/n^{1/5})$.

If $A(S)$ occurs, then (3.9) holds, and we define the sets $T_1, T_2, \ldots, \hat{T}_1, \hat{T}_2, \ldots$ as we defined $S_1, S_2, \ldots, \hat{S}_1, \hat{S}_2, \ldots$ above. This time there is no conditioning on the points

not in S and so for a given t we have, by Lemma 3.1 with $\epsilon = 1/2$

$$\text{(3.11)} \qquad \Pr\left(\left|\{j \colon X_t \in T_t\}\right|\right.$$
$$\left.\leqslant (\beta' - \alpha)^m \delta(n-m)/2\right) \leqslant \exp\left(-(\beta' - \alpha)^m \delta(n-m)/8\right)$$
$$\leqslant \exp\left(-(m+5)\log n\right) \quad \text{for } n \text{ large.}$$

By summing over all possible values for S and $t = 1, 2, \ldots, L$ we obtain (3.10).

Suppose now $W_j \in R^n$ is defined by $X_j = (W_j, p_j)$ for $j = 1, 2, \ldots, n$. Then because the whole of $\hat{S}_t$ projects down onto a rectangular region B in the hyperplane $p = 0$ and because of Lemma 2.1 we know that

$$\text{(3.12)} \qquad \left\{W_j \colon X_j \in \hat{S}_t\right\}$$

are independently and uniformly distributed over B.

We will therefore restrict our attention to $X_j \in \hat{S}_t$ in our search for T satisfying (3.5). We now define

$$w'_{ij} = \left(w_{ij} - \tfrac{1}{2}(\beta' + \alpha)\right)\sqrt{12}/(\beta' - \alpha) \quad \text{for } X_j \in S_t.$$

Then by (3.12) the w'_{ij} are independent and uniformly distributed in $[-\sqrt{3}, \sqrt{3}]$ and hence have mean 0 and variance 1. This will allow us to use Lemma 3.4 below.

Now let N_t denote some set of $2k$ indices j with $X_j \in S_t$. By (3.10) we can find such a set for all $t = 1, 2, \ldots, L$ with probability exceeding $1 - O(1/n^4)$.

Conditional on this we show that *independently*, for $t = 1, 2, \ldots, L$,

$$\text{(3.13)} \quad \Pr\Bigg(\exists K_t \subseteq N_t \text{ such that}$$
$$\text{(i) } |K_t| = k,$$
$$\left.\text{(ii) } \sum_{j \in K_t} w'_{ij} \in [c'_i - \theta', c'_i] \text{ for } i = 1, 2, \ldots, m\right) \geqslant \frac{1 + o(1)}{1 + (2/\sqrt{3})^m} \quad \text{where}$$

$$c'_i = \frac{\sqrt{12}}{\beta' - \alpha}\left(c_i - \frac{k}{2}(\beta' + \alpha)\right) \quad \text{and}$$

$$\theta' = \frac{\sqrt{12}}{\beta' - \alpha}\theta = \left(\frac{(\pi k)^{(m+1)/2}}{4^k}\right)^{1/m}\sqrt{2}.$$

Note that

$$|c'_i| \leqslant \frac{\sqrt{12}}{\beta - \alpha}\max(m, \alpha k/2) = O(1)$$

as m is constant. But (3.13) follows directly from the following Lemma. It generalizes a lemma of Lueker [8]. The proof is rather long and messy and is given in an appendix.

LEMMA 3.4. *Let Y_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, 2k$ be independent, identically distributed random variables in the range $[-a, a]$. Let $E(Y_{11}) = 0$ and* $\operatorname{Var}(Y_{11}) = 1$ *and* *suppose* Y_{11} *has a continuous density* g *satisfying* $g(x) \leqslant M$. *Given* m *intervals* $I_i = [A_i - \theta, A_i + \theta]$ *where* $\theta = ((\pi k)^{(m+1)/2}/4^k)^{1/m}/\sqrt{2}$ *and* $A_i = O(k^\alpha)$ *where* $\alpha < 1/2$

$$\Pr\left(\exists K \subseteq \{1,2,\dots,2k\} : |K| = k,\ \sum_{j \in K} Y_{ij} \in I_i,\ i = 1,2,\dots,m\right) \geqslant \frac{1 + o(1)}{1 + (2/\sqrt{3})^m}$$

assuming $k \to \infty$. ∎

The proof of (a) is now complete.

(b) If E_t occurs then, in the notation of Lemma 3.2, we consider the solution $x_j = 1$ if and only if $j \in N' \cup T$. We then have

$$\sum_{j \in N' \cup T} w_{ij} \in \begin{cases} [b_i - \theta, b_i], & i \in M^*, \\ [b_i - (b_i' - b_i'') - \theta, b_i - (b_i' - b_i'')], & i \notin M^*. \end{cases}$$

We then have only to substitute the following inequalities into (3.3) to obtain the result

$$\bar{p}_j \geqslant -t\delta, \qquad j \in T,$$

$$\delta_i \leqslant \theta, \qquad i \in M^*. \quad ∎$$

The proof of Theorem 1.1 is essentially complete. (1.5) follows from Lemma 3.3 and (1.4) is a consequence of (1.5).

4. Proof of the Theorems 1.2 and 1.3. In this section we prove Theorems 1.2 and 1.3. In the following a_2, a_3 are as in (1.5). Let us first describe

Algorithm A_ϵ.

(0) $$t_0 := \left\lceil \frac{\log \epsilon}{\log(1 - a_3)} \right\rceil + 1.$$

(1) Solve LP(b);
let $\mathbf{x}^*$ denote the optimal basic feasible solution;
let $S_i = \{ j \in V_n : x_j^* = i \}$, $i = 0$ or 1 and
$S = \{ j \in V_n : x_j \text{ is basic} \}$.
(Note that $S = Z_n - (S_0 \cup S_1)$ with probability one.)

(2) Let $T_0 = \{ j \in V_n : |\bar{p}_j| \leqslant t_0 a_2((\log n)^2/n) \}$ where $\bar{p}_j$ is the reduced cost of x_j—see (2.1).

(3) For each $Y \subseteq V_n$ satisfying

(4.1a) $$T = (Y \cap S_0) \cup (S_1 - Y) \subseteq T_0,$$

(4.1b) $$\sum_{j \in T} |\bar{p}_j| \leqslant t_0 a_2 \frac{(\log n)^2}{n},$$

check feasibility i.e. check if $\sum_{j \in Y} w_{ij} \leqslant b_i$, $i = 1,\dots,m$.

(4) If Step 3 finds any feasible solutions, let $\hat{Y}$ be the one found with the largest objective value, and let $\hat{V}$ be this value.
If

$$\hat{V} \geqslant V_{\mathrm{LP}} - t_0 a_2\left((\log n)^2/n\right)$$

then output $\hat{Y}$ is optimal
else output $\hat{Y}$ is approximate.
END

We must show that if Step 4 claims Y to be optimal, then it is, and that this occurs with probability exceeding $1 - \epsilon$. Also we must show that the algorithm runs in polynomial time.

Thus let us consider any feasible solution $\mathbf{x}$ to IP and let $Y = \{ j \in V_n : x_j = 1 \}$. Basic linear programming theory gives that

$$\sum_{j \in Y} p_j \leqslant V_{\mathrm{LP}} - \sum_{j \in (Y \cap S_0) \cup (S_1 - Y)} |\bar{p}_j|.$$

We thus have

$$\sum_{j \in Y} p_j < V_{\mathrm{LP}} - t_0 a_2 \frac{(\log n)^2}{n}$$

unless (4.1) holds.

Thus if Step 4 declares Y to be optimal, it is.

Now Theorem 1.1 shows that

$$\Pr\left(V_{\mathrm{LP}} - V_{\mathrm{IP}} \geqslant t_0 a_2 \frac{(\log n)^2}{n} \right) \leqslant \epsilon(1 - a_3)$$

and so the probability that A_ϵ fails to find the optimal solution is no more than $(1 - a_3)\epsilon$.

We must now consider A_ϵ's execution time. Thus let $W = \{ Y \subseteq V_n : (4.1) \text{ holds} \}$. We show next that

(4.2) $$\Pr\left(|W| \geqslant n^{d_0(\epsilon, m, \beta)}\right) = O(1/n)$$

for some $d_0(\epsilon, m, \beta) > 0$.

Let $a = (m + 2)/(3 \log(3/e))$ and $\delta = a \log n / n$.

Let the S_t for integer t (positive or negative) be defined relative to $H(\mathbf{b})$ as they were relative to $H(\mathbf{b}')$ prior to Lemma 3.3. For each t let $N_t = \{ j \in Z_n : X_j \in S_t \}$. We prove

(4.3) $$\Pr\left(\exists t : |t| \leqslant (t_0 a_2 \log n)/a + 1 \text{ and } |N_t| \geqslant 3a \log n\right) = O(1/n).$$

Because of conditioning problems, we have to proceed as in the proof of (3.10) and consider all $O(n^m)$ possible sets of basic variables and their associated hyperplanes.

So let us fix a set of m basic variables S, the hyperplane H through X_j: $j \in S$ and integer t and the set S_t defined relative to H.

By projecting S_t down onto the face $p = 0$ of C we see that $\mathrm{Vol}(S_t) \leqslant \delta$. Thus

$$\begin{aligned} \Pr(|N_t| \geqslant 3a \log n) &\leqslant \binom{n - m}{\lceil 3a \log n \rceil} \delta^{\lceil 3a \log n \rceil} \\ &\leqslant \left(\frac{ne\delta}{\lceil 3a \log n \rceil} \right)^{\lceil 3a \log n \rceil} \\ &= O(n^{-(m+2)}). \end{aligned}$$

(4.3) follows immediately.

We now follow the general line of the argument of [5]. Let

$$R_i = \bigcup_{r=2^{i-1}}^{2^i-1} S_{r+1},$$

$$R_{-i} = \bigcup_{r=2^{i-1}}^{2^i-1} S_{-r}, \qquad i = 1,2,\ldots,i_{\max}\lceil \log_2(t_0 a_2 \log n/a)\rceil, \quad \text{and}$$

$$R_0 = S_0 \cup S_1.$$

For a given T in (4.1a) there are at most 2^m sets $Y \subseteq Z_n$ giving this T. As m is constant we need only estimate the number of T satisfying (4.1b). So assume (4.1b) holds and let $n_i = |\{j \in T\colon X_j \in R_i\}|$. Then (4.1b) implies $n_i \leqslant \alpha_i$, where $\alpha_i = \lfloor 2^{1-|i|} t_0 a_2 \log n/a \rfloor$ for $i \neq 0$.

Let now $\beta_i = \lfloor 3 \times 2^{|i|-1} \times a \log n \rfloor$. Then it follows from (4.3) that

$$\text{(4.4)} \qquad \Pr\left(|W| \geqslant 2^{m+2\beta_0} \prod_{i=1}^{i_{\max}} \lambda_i^2\right) = O(1/n) \quad \text{where}$$

$$\lambda_i = \begin{cases} 2^{\beta_i} & \text{for } 2^{2(i-1)} < t_0 a_2/a^2, \\ \sum_{r=0}^{\alpha_i} \binom{\beta_i}{r} & \text{otherwise.} \end{cases}$$

Here we are estimating $|W|$ by the product of the number of subsets of R_i, $i \leqslant i_{\max}$, which have cardinality no more than α_i.

In the second alternative, denoted by $i \geqslant i_0$ say, $\alpha_i \leqslant (\beta_i + 1)/3$ which implies $\lambda_i \leqslant \binom{\beta_i}{\alpha_i+1}$ for n large. Thus

$$\prod_{i=1}^{i_{\max}} \lambda_i^2 \leqslant 2^{6\sqrt{t_0 a_2}\log n} \prod_{i=i_0}^{i_{\max}} ((\beta_i e)/(\alpha_i + 1))^{2(\alpha_i+1)}$$

$$\leqslant 2^{6\sqrt{t_0 a_2}\log n} \prod_{i=i_0}^{i_{\max}} (3a^2 4^{i-1} e/a_2 t_0)^{2(2^{1-i} t_0 a_2 \log n/a+1)}$$

$$= O(n^{d_1(\epsilon, m, \beta)})$$

for some $d_1(\epsilon, m, \beta)$, as $\sum_{i=0}^{\infty}(i/2^i)$ converges and $i_{\max} = O(\log\log n)$. This implies (4.2).

Our model of computation is a real RAM as in, for example, [3], so it should be clear from the proof of (4.3) that Step 3 runs in polynomial time with probability $1 - O(1/n)$. We of course have the option of stopping after $cn^{d(\epsilon,m,\beta)}$ subsets, for some suitable constant c, have been enumerated, if we have not exhausted W.

The probability that A_ϵ fails to solve IP is then $\epsilon(1 - a_2) + O(1/n)$ and Theorem 1.2 follows. ∎

Since m is fixed, Step 2 can be executed in $O(n)$ time—see Dyer [3].

We turn now to the proof of Theorem 1.3. Consider the solution to LP(b). Let $T = \{j \in V_n\colon |\bar{p}_j| \leqslant \log n/n\}$. Then

$$\text{(4.5)} \qquad MAXCH \leqslant (V_{\mathrm{LP}} - V_{\mathrm{IP}})n/\log n + |T|$$

where the first term bounds the number of 'changes outside T' and the second term bounds the number of changes inside T.

Now Theorem 1.1 yields

$$\text{(4.6)} \qquad \Pr\left(V_{\mathrm{LP}} - V_{\mathrm{IP}} \geqslant \frac{t}{2}\frac{(\log n)^2}{n}\right) \leqslant (1 - a_3)^{t/2a_2}$$

and using the same technique as was used to prove (3.9) and (4.3) we have

$$\text{(4.7)} \quad \Pr\left(|T| \geqslant \frac{t\log n}{2}\right) = O\left(\binom{n+m}{m}\binom{n-m}{\left\lceil \frac{t\log n}{2} - m\right\rceil}\left(\frac{2\log n}{n}\right)^{\lceil t\log n/2 - m\rceil}\right)$$

$$= O\left(n^m\left(\frac{ne}{\frac{t\log n}{2} - m}\right)^{\lceil t\log n/2-m\rceil}\left(\frac{2\log n}{n}\right)^{\lceil t\log n/2-m\rceil}\right)$$

$$= O\left(n^{m-(t/2)\log(t/4e)}(t/4e)^m\right).$$

The result now follows from (4.5), (4.6) and (4.7). ∎

Acknowledgement. It is a pleasure to acknowledge the detailed and constructive comments of one of the referees. In particular we would like to thank him/her for the strengthening of Lemma 3.4.

Appendix 1.

PROOF OF LEMMA 3.4. Let G_k, g_k denote the probability and density functions of the sum of k independent Y_{ij}'s. Define $\bar{G}(x) = G(x\sqrt{k})$ and $\bar{g}(x) = g(x\sqrt{k})\sqrt{k}$. For $K \subseteq \{1,2,\ldots,2k\}$, $|K| = k$, let

$$\delta_K = 1 \quad \text{if } \sum_{j\in K} Y_{ij} \in I_i, \qquad i = 1,\ldots,m,$$

$$= 0 \quad \text{otherwise.}$$

Let $Z = \sum_K \delta_K$. We use the following standard inequality to prove the lemma

$$\text{(A1)} \qquad \Pr(Z > 0) \geqslant E(Z)^2/E(Z^2).$$

Then

$$E(Z) = \binom{2k}{k}\prod_{i=1}^m (G_k(A_i + \theta) - G_k(A_i - \theta))$$

$$= \binom{2k}{k}\prod_{i=1}^m \left(\bar{G}_k\left(\frac{A_i+\theta}{\sqrt{k}}\right) - \bar{G}_k\left(\frac{A_i-\theta}{\sqrt{k}}\right)\right)$$

$$= \binom{2k}{k}\prod_{i=1}^m \bar{g}_k\left(\frac{A_i'}{\sqrt{k}}\right)\cdot\frac{2\theta}{\sqrt{k}} \quad \text{for some } A_i' \in I_i \text{ (Rolle's Theorem)}$$

$$= \binom{2k}{k}\left(\frac{2\theta}{\sqrt{k}}\right)^m \prod_{i=1}^m \left(\phi\left(\frac{A_i'}{\sqrt{k}}\right) + O\left(\frac{1}{\sqrt{k}}\right)\right)$$

(where $\phi(x) = e^{-x^2/2}/\sqrt{2\pi}$ and we have used Theorem XVI.2.1 of Feller [4])

$$= \binom{2k}{k}\left(\frac{2\theta}{\sqrt{2\pi k}}\right)^m (1 + o(1)) \quad \text{since } A_i' = o(\sqrt{k}).$$

Next let $S_i = \{i, i+1, \ldots, i+k-1\}$, $i = 1, 2, \ldots, k+1$.
Then

$$E(Z^2) = \sum_K \sum_{K'} E(\delta_K \delta_{K'})$$

$$= \binom{2k}{k}^2 \sum_{r=0}^{k} \frac{\binom{k}{r}\binom{k}{k-r}}{\binom{2k}{k}} E(\delta_{S_1}\delta_{S_{r+1}}).$$

Case 1. $r = 0$: $E(\delta_{S_1}^2) = E(\delta_{S_1}) = (2\theta/\sqrt{2\pi k})^m(1 + o(1))$.

Case 2. $r \geqslant 1$: Let $U_i = \Sigma_{j=1}^r Y_{ij}$, $V_i = \Sigma_{j=r+1}^k Y_{ij}$ and $W_i = \Sigma_{j=k+1}^{k+r} Y_{ij}$. Then

$$E(\delta_{S_1}\delta_{S_{r+1}}) = \prod_{i=1}^m \left(\int_{V_i} g_{k-r}(V_i)(G_r(A_i - V_i + \theta) - G_r(A_i - V_i - \theta))^2 \, dV_i\right)$$

$$= 4^m \theta^{2m} \prod_{i=1}^m \int_{V_i} g_{k-r}(V_i) g_r(A_i' - V_i)^2 \, dV_i$$

where $A_i' \in [A_i - \theta, A_i + \theta]$ by Rolle's Theorem.

Now $g(x) \leqslant M$ $(\forall x)$ implies $g_t(x) \leqslant M$ $(\forall x, t)$. This can be proved by induction. We thus have $E(\delta_{S_1}\delta_{S_{r+1}}) \leqslant 4^m\theta^{2m}M^{2m}$. For $|r - k/2| \leqslant \sqrt{k}\log k$, we choose $1/2 < \beta < 1$ and consider

$$\int_{-\infty}^{\infty} g_{k-r}(V_i) g_r(A_i' - V_i)^2 \, dV_i = \int_{-\infty}^{-k^\beta} + \int_{-k^\beta}^{k^\beta} + \int_{k^\beta}^{\infty} \quad \text{in an obvious notation.}$$

Now

$$\int_{k^\beta}^{\infty} \leqslant M^2 \Pr\left(\sum_{j=1}^{k-r} Y_{ij} \geqslant k^\beta\right)$$

$$\leqslant M^2 \exp\{-(1 + o(1))k^{2\beta-1}/a^2\},$$

$$= o(k^{-s}) \quad \text{for any } s > 0, \text{ since } \beta > 1/2,$$

using Theorem 2 of Hoeffding [6], with

$$n \leftarrow (k - r) = \tfrac{1}{2}k(1 + o(1)), \quad \mu \leftarrow 0, \quad t \leftarrow k^\beta/(k - r), \quad a_i \leftarrow -a, \quad b_i \leftarrow a.$$

The same inequality holds for $\int_{-\infty}^{-k^\beta}$.

$$\int_{-k^\beta}^{k^\beta} = \int_{-k^\beta}^{k^\beta} \bar{g}_{k-r}\left(\frac{V_i}{\sqrt{k-r}}\right) \cdot \frac{1}{\sqrt{k-r}} \cdot \bar{g}_r\left(\frac{A_i' - V_i}{\sqrt{r}}\right)^2 \cdot \frac{1}{r} \cdot dV_i$$

$$= \frac{1}{r\sqrt{k-r}} \int_{-k^\beta}^{k^\beta} \left(\phi\left(\frac{V_i}{\sqrt{k-r}}\right) + O(k^{-1})\right)\left(\phi\left(\frac{A_i - V_i}{\sqrt{r}}\right)^2 + O(k^{-1})\right) dV_i$$

using Feller [4] as before

$$= \frac{1}{r\sqrt{k-r}}\left(\int_{-k^\beta}^{k^\beta} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{V_i^2}{2(k-r)}\right) \cdot \frac{1}{2\pi}\exp\left(-\frac{(A_i - V_i)^2}{r}\right) dV_i + O(k^{\beta-1})\right)$$

$$= \frac{1}{(2\pi)^{3/2} r\sqrt{k-r}}\left(\int_{-k^\beta}^{k^\beta} \exp\left(-\frac{1}{2}\left(\left(\frac{1}{k-r} + \frac{2}{r}\right) \cdot \left(V_i - A_i\Big/\left(1 + \frac{r}{2(k-r)}\right)\right)^2 + \frac{2}{2k-r}A_i^2\right)\right) dV_i + O(k^{\beta-1})\right)$$

$$= \frac{1 + o(1)}{(2\pi)^{3/2} r\sqrt{k-r}}\left(\int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}\left(\frac{1}{k-r} + \frac{2}{r}\right) \cdot \left(V_i - A_i\Big/\left(1 + \frac{r}{2(k-r)}\right)\right)^2\right) dV_i + O(k^{\beta-1})\right)$$

$$= \frac{1 + o(1)}{(2\pi)^{3/2} r\sqrt{k-r}}\left(\sqrt{2\pi\Big/\left(\frac{1}{k-r} + \frac{2}{r}\right)} + O(k^{\beta-1})\right)$$

$$= \frac{1 + o(1)}{2\pi\sqrt{r(2k-r)}}$$

$$= \frac{1 + o(1)}{\pi k\sqrt{3}} \quad \text{since } \left|r - \frac{k}{2}\right| \leqslant \sqrt{k}\log k.$$

Now Hoeffding [6] shows that

$$\sum_{|r-k/2| \leqslant \sqrt{k}\log k} \frac{\binom{k}{r}\binom{k}{k-r}}{\binom{2k}{k}} = 1 - o(k^{-p}), \quad \text{for any } p > 0.$$

It follows that

$$E(Z^2) = (1 + o(1))\left(\frac{2\theta}{\sqrt{2\pi k}}\right)^m \binom{2k}{k} + \binom{2k}{k}^2 4^m\theta^{2m}M^{2m}o(k^{-m})$$

$$+ (1 + o(1))\binom{2k}{k}^2\left(\frac{4\theta^2}{\pi k\sqrt{3}}\right)^m (1 - o(1)).$$

The result now follows from (A1) after some algebra.

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FRIEZE: CARNEGIE-MELLON UNIVERSITY AND QUEEN MARY COLLEGE

MATHEMATICS OF OPERATIONS RESEARCH
Vol. 14, No. 1, February 1989
Printed in U.S.A.

COERCION FUNCTIONS AND DECENTRALIZED LINEAR PROGRAMMING*†

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This paper describes a decentralized linear programming solution procedure. Unlike the classic Dantzig-Wolfe decomposition algorithm, this procedure is a complete decentralization, where the subproblems act independently to generate the optimal solution. In Dantzig-Wolfe decomposition, the master problem must "dictate" the final optimal solution. Our procedure is an improvement over Dantzig-Wolfe decomposition in this sense. The procedure works by adding a "coercion" function to the subproblem objective. Under proper conditions, this function "coerces" the subproblem to behave as desired. Such total decentralization is important for loosely coupled systems where it is not possible for the master level to dictate optimal decision values for the subsystems.

1. Introduction. This paper describes a decentralized linear programming solution procedure. Unlike the classic result of Dantzig and Wolfe [DANT61], our procedure is a complete decentralization. In the Dantzig-Wolfe decomposition, a master problem generates "prices" and subproblems generate feasible solutions to the overall problem. However, when the procedure converges, the master problem must dictate the optimal decision values to each subproblem. Dantzig refers to this as "central planning without complete information at the center." [DANT63]. Due to the linearity of the subproblems, there is usually no set of prices that will induce the subproblems to return the optimal decision values.

Our procedure uses the idea of a coercion function, a function that is added to the subproblem objective. As the name implies, a coercion function is designed to coerce the subproblem to return the optimal solution. Using a coercion function, the solution procedure can be completely decentralized.

For our purpose, we will parameterize the coercion function with a vector α. This vector will serve the role played by dual prices in the Dantzig-Wolfe decomposition. At each iteration of our procedure, the subproblems will generate a new feasible solution for the overall problem. The master problem uses this "proposed solution" to generate new values for α. We will develop conditions under which this procedure converges to the optimal solution for the overall linear program.

2. Characterizing coercion functions. Consider the following dual linear programs:

Primal Problem (P)	Dual Problem (D)
$\min \ \mathbf{cx}$	$\max \ [\phi, \Pi]\begin{bmatrix} \mathbf{b}_0 \\ \mathbf{b}_1 \end{bmatrix}$
s.t. $\mathbf{Ax} \geqslant \mathbf{b}_0,$	s.t. $[\phi, \Pi]\begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix} \leqslant \mathbf{c},$
$\mathbf{Bx} \geqslant \mathbf{b}_1,$	
$\mathbf{x} \geqslant 0.$	$\phi, \Pi \geqslant 0.$

*Received 21, February 1986; revised October 8, 1987.
AMS 1980 subject classification. Primary: 90C05. Secondary: 90C27.
IAOR 1973 subject classification. Main: Programming: Linear.
OR/MS Index 1978 subject classification. Primary: 643 Linear, Programming Algorithms.
Key words. Decentralization, distributed computing, linear programming, decomposition.
†This work was sponsored in part by National Science Foundation Grant #51496.

0364-765X/89/1401/0177$01.25
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