Long paths in random Apollonian networks

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February 6, 2014

Abstract

We consider the length $L(n)$ of the longest path in a randomly generated Apollonian Network (ApN) $A_n$. We show that w.h.p. $L(n) \leq ne^{-\log^c n}$ for any constant $c < 2/3$.

1 Introduction

This paper is concerned with the length of the longest path in a random Apollonian Network (ApN) $A_n$. We start with a triangle $T_0 = xyz$ in the plane. We then place a point $v_1$ in the centre of this triangle creating 3 triangular faces. We choose one of these faces at random and place a point $v_2$ in its middle. There are now 5 triangular faces. We choose one at random and place a point $v_3$ in its centre. In general, after we have added $v_1, v_2, \ldots, v_n$ there will $2n + 1$ triangular faces. We choose one at random and place $v_n$ inside it. The random graph $A_n$ is the graph induced by this embedding. It has $n + 3$ vertices and $3n + 6$ edges.

This graph has been the object of study recently. Frieze and Tsourakakis [4] studied it in the context of scale free graphs. They determined properties of its degree sequence, properties of the spectra of its adjacency matrix, and its diameter. Cooper and Frieze [2], Ebrahimzadeh, Farczadi, Gao, Mehrabian, Sato, Wormald and Zung [3] improved the diameter result and determine the diameter asymptotically. The paper [3] proves the following result concerning the length of the longest path in $A_n$:

**Theorem 1** There exists an absolute constant $\alpha$ such that if $L(n)$ denotes the length of the longest path in $A_n$ then
\[
\Pr \left( L(n) \geq \frac{n}{\log^\alpha n} \right) \leq \frac{1}{\log^\alpha n}.
\]

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The value of $\alpha$ from [3] is rather small and we will assume for the purposes of this proof that

$$\alpha < \frac{1}{3}. \quad (1)$$

The aim of this paper is to give the following improvement on Theorem 1:

**Theorem 2**

$$\Pr(L(n) \geq ne^{-\log^c n}) \leq O(e^{-\log^{c/2} n})$$

for any constant $c < 2/3$.

This is most likely far from the truth. It is reasonable to conjecture that in fact $L(n) \leq n^{1-\varepsilon}$ w.h.p. for some positive $\varepsilon > 0$. For lower bounds, [3] shows that $L(n) \geq n^{\log_3 2} + 2$ always and $E(L(n)) = \Omega(n^{0.8})$. Chen and Yu [1] have proved an $\Omega(n^{\log_3 2})$ lower bound for arbitrary 3-connected planar graphs.

2 Outline proof strategy

We take an arbitrary path $P$ in $A_n$ and bound its length. We do this as follows. We add vertices to the interior of $xyz$ in rounds. In round $i$ we add $\sigma_i$ vertices. We start with $
abla = n^{1/2}$ and choose $\sigma_i \gg \sigma_{i-1}$ where $A \gg B$ iff $B = o(A)$. We will argue inductively that $P$ only visits $\tau_{i-1} = o(\sigma_{i-1})$ faces of $A_{\sigma_{i-1}}$ and then use Lemma 2 below to argue that roughly a fraction $\tau_{i-1}/\sigma_{i-1}$ of the $\sigma_i$ new vertices go into faces visited by $P$. We then use a variant (Lemma 3) of Theorem 1 to argue that w.h.p. $\frac{\tau_i}{\sigma_i} \leq \frac{\tau_{i-1}}{2\sigma_{i-1}}$. Theorem 2 will follow easily from this.

3 Paths and Triangles

Fix $1 \leq \sigma \leq n$ and let $A_\sigma$ denote the ApN we have after inserting $\sigma$ vertices $A$ interior to $T_0$. It has $2\sigma + 1$ faces, which we denote by $T = \{T_1, T_2, \ldots, T_{2\sigma+1}\}$. Now add $N$ more vertices $B$ to create a larger network $A'_{\sigma'}$ where $\sigma' = \sigma + N$. Now consider a path $P = x_1, x_2, \ldots, x_m$ through $A'_{\sigma'}$. Let $I = \{i : x_i \in A\} = \{i_1, i_2, \ldots, i_\tau\}$. Note that $Q = (i_1, i_2, \ldots, i_\tau)$ is a path of length $\tau - 1$ in $A_\sigma$. This is because $i_1i_{k+1}, 1 \leq k < \tau$ must be an edge of some face in $T$. We also see that for any $1 \leq k < \tau$ that the vertices $x_j, i_k < j < i_{k+1}$ will all be interior to the same face $T_l$ for some $l \in [2\sigma + 1]$.

We summarise this in the following lemma: We use the notation of the preceding paragraph.
Lemma 1 Suppose that $1 \leq \sigma < \sigma' \leq n$ and that $Q$ is a path of $A_\sigma$ that is obtained from a path $P$ in $A_{\sigma'}$ by omitting the vertices in $B$.

Suppose that $Q$ has $\tau$ vertices and that $P$ visits the interior of $\tau'$ faces from $T$. Then

$$\tau - 1 \leq \tau' \leq \tau + 1.$$

Proof The path $P$ breaks into vertices of $A_\sigma$ plus $\tau + 1$ intervals where in an interval it visits the interior of a single face in $T$. This justifies the upper bound. The lower bound comes from the fact that except for the face in which it starts, if $P$ re-enters a face $xyz$, then it cannot leave it, because it will have already visited all three vertices $x, y, z$. Thus at most two of the aforementioned intervals can represent a repeated face.

4 A Structural Lemma

Let

$$\lambda_1 = \log^2 n.$$

Lemma 2 The following holds for all $i$. Let $\sigma = \sigma_i$ and suppose that $\lambda_1 \leq \tau \ll \sigma$. Suppose that $T_1, T_2, \ldots, T_{\tau}$ is a set of triangular faces of $A_\sigma$. Suppose that $N \gg \sigma$ and that when adding $N$ vertices to $A_\sigma$ we find that $M_j$ vertices are placed in $T_j$ for $j = 1, 2, \ldots, \tau$. Then for all $J \subseteq [2\sigma + 1], |J| = \tau$ we have

$$\sum_{j \in J} M_j \leq \frac{100\tau N}{\sigma} \log \left( \frac{\sigma}{\tau} \right).$$

This holds q.s.\(^1\) for all choices of $\tau, \sigma$ and $T_1, T_2, \ldots, T_{\tau}$.

Proof We consider the following process. It is a simple example of a branching random walk. We consider a process that starts with $s$ newly born particles. Once a particle is born, it waits an exponentially mean one distributed amount of time. After this time, it simultaneously dies and gives birth to $k$ new particles and so on. A birth corresponds to a vertex of our network and a particle corresponds to a face.

Let $Z_t$ denote the number of deaths up to time $t$. The number of particles in the system is $\beta_N = s + N(k - 1)$. Then we have

$$\Pr(Z_{t+dt} = N) = \beta_{N-1} \Pr(Z_t = N-1)dt + (1 - \beta_N dt) \Pr(Z_t = N).$$

\(^1\)A sequence of events $\mathcal{E}_n$ holds quite surely (q.s.) if $\Pr(\neg \mathcal{E}_n) = O(n^{-K})$ for any constant $K > 0.$
So, if \( p_N(t) = \Pr(Z_t = N) \), we have \( f_N(0) = 1_{N=s} \) and

\[
p_N'(t) = \beta_{N-1}p_{N-1}(t) - \beta_N p_N(t).
\]

This yields

\[
p_N(t) = \prod_{i=1}^{N} \frac{(k-1)(i-1) + s}{(k-1)i} \times e^{-st}(1 - e^{-(k-1)t})^N
\]

\[
= A_{k,N,s} e^{-st}(1 - e^{-(k-1)t})^N.
\]

\( A_{3,0,s} = 1 \). When \( s \) is even, \( s, N \to \infty \), and \( k = 3 \) we have

\[
A_{3,N,s} = \prod_{i=1}^{N} \left( \frac{s/2 + i - 1}{i} \right),
\]

\[
\approx \left( 1 + \frac{s-2}{2N} \right)^N \left( 1 + \frac{2N}{s-2} \right)^{s/2-1} \sqrt{\frac{2N+s}{2\pi N s}}.
\]

We also need to have an upper bound for small even \( s, N^2 = o(s) \), say. In this case we use

\[
A_{3,N,s} \leq s^N.
\]

When \( s \geq 3 \) is odd, \( s, N \to \infty \) (no need to deal with small \( N \) here) and \( k = 3 \) we have

\[
A_{3,N,s} = \prod_{i=1}^{N} \left( \frac{2i - 2 + s}{2i} \right) = \frac{(s-1+2N)!((s-1)/2)!}{2^2N(s-1)!N!(s-1/2+N)!}
\]

\[
\approx \left( 1 + \frac{s-1}{2N} \right)^N \left( 1 + \frac{2N}{s-1} \right)^{(s-1)/2} \frac{1}{(2\pi N)^{1/2}}.
\]

We now consider with \( \tau \to \infty, \tau \ll \sigma, N \geq m \geq 2\tau N/\sigma \gg \tau \) and arbitrary \( t \),

(under the assumption that \( \tau \) is odd and \( \sigma \) is odd)
Suppose first that

\[ \Pr(M_1 + \cdots + M_\tau = m \mid M_1 + \cdots + M_\sigma = N) \]

\[ = \frac{\Pr(M_1 + \cdots + M_\tau = m) \Pr(M_{\tau+1} + \cdots + M_\sigma = N - m)}{\Pr(M_1 + \cdots + M_\sigma = N)} \]

\[ = \frac{A_{3,m,\tau} A_{3,N-m,\sigma-\tau}}{A_{3,N,\sigma}} \]

\[ \approx \frac{(1 + \frac{\tau - 1}{2m})(1 + \frac{2m}{\tau - 1})^{(\tau - 1)/2} \left(1 + \frac{\sigma - \tau - 2}{2(N - m)}\right)^{N-m} \left(1 + \frac{2(N - m)}{\sigma - \tau - 2}\right)^{(\sigma - \tau - 2)/2} (N(2(N - m) + \sigma))^{1/2}}{(1 + \frac{\sigma - 1}{2N})^N \left(1 + \frac{2N}{\sigma - 1}\right)^{(\sigma - 1)/2} (2\pi m \sigma (N - m))^{1/2}} \]

(2)

The above bound can be re-written as

\[ \leq_b \frac{e^{o(\tau)} \left(\frac{2m}{\tau}\right)^{(\tau - 1)/2} N^{1/2} \sigma^{(\sigma - 1)/2}}{(2N)^{\sigma - 1/2} \sigma^{1/2}} \times \frac{m^{(\tau - 1)/2} \left(1 + \frac{2(N - m)}{\sigma - \tau - 2}\right)^{(\sigma - \tau - 2)/2} (N - m + \sigma)^{1/2}}{(m(N - m))^{1/2}}. \]

Suppose first that \( m \leq N - 4\sigma \). Then the bound becomes

\[ \leq_b \frac{e^{o(\tau)} \left(\frac{2m}{\tau}\right)^{(\tau - 1)/2} N^{1/2} \sigma^{(\sigma - 1)/2}}{(2N)^{\sigma - 1/2} \sigma^{1/2}} \times \frac{m^{(\tau - 2)/2} \left(1 + \frac{2(N - m)}{\sigma - \tau - 2}\right)^{(\sigma - \tau - 2)/2} (N - m + \sigma)^{1/2}}{(m(N - m))^{1/2}} \]

(3)

We inflate this by \( n^2 \left(\frac{2\sigma + 1}{\tau}\right) \) to account for our choices for \( \tilde{\sigma}, \tilde{\tau}, T_1, \ldots, T_\tau \) to get

\[ \leq_b n^2 \frac{e^{o(\tau)} N^{1/2}}{m^{1/2}} \left(\frac{e^2 m \sigma}{\tau N} \cdot \exp \left\{ - m \sigma (\frac{\tau - 1}{m} + \frac{2\sigma^2}{(\tau - 1)(N - m)} \right\} \right)^{(\tau - 1)/2}. \]
So, if \( m_0 = \frac{100rN\log(\sigma/\tau)}{\sigma} \) then

\[
\sum_{m=m_0}^{N-4\sigma} \Pr(\exists \sigma, \tau, T_1, \ldots, T_\tau : M_1 + \cdots + M_\tau = m \mid M_1 + \cdots + M_\sigma = N) \\
\leq b n^2 e^{o(\tau)} N^{5/2} \sum_{m=m_0}^{N-4\sigma} \left( \frac{4e^4m_0\sigma^3}{\tau^3N} \cdot \exp\left\{-\frac{m_0\sigma}{3\tau N}\right\} \right)^{(\tau-1)/2} \\
\leq \frac{n^2}{\tau N} \sum_{m=m_0}^{N-4\sigma} \left( \frac{4e^4m_0\sigma^3}{\tau^3N} \cdot \exp\left\{-\frac{m_0\sigma}{3\tau N}\right\} \right)^{(\tau-1)/2}
\]

since \( xe^{-Ax} \) is decreasing for \( Ax \geq 1 \)

\[
= n^2 e^{o(\tau)} N^{7/2} \left( \frac{4e^4m_0\sigma^3}{\tau N} \cdot \exp\left\{-\frac{m_0\sigma}{6\tau N}\right\} \right)^{(\tau-1)/2} \\
\leq n^2 N^{7/2} \left( 400e^{4+o(1)} \log\left(\frac{\sigma}{\tau}\right) \cdot e^{-50/3} \cdot \exp\left\{-\frac{m_0\sigma}{3\tau N}\right\} \right)^{(\sigma-1)/2} \\
= O(n^{-\text{any constant}}).
\]

Suppose now that \( N - 4\sigma \leq m \leq N - \sigma^{1/3} \). Then we can bound (3) by

\[
\leq b n^2 \frac{e^{o(\tau)} \left(\frac{2}{\tau}\right)^{\tau-1/2} \cdot (2N)^{\sigma-1/2}}{\sigma^{1/2} \cdot \tau^\sigma} \cdot m^{(\sigma-1)/2} e^{4\sigma} \\
\leq \left(\frac{e^8 \sigma}{2N}\right)^{(\sigma-1)/2} \left(\frac{e^8 \sigma}{\tau}\right)^{(\tau-1)/2}.
\]

We inflate this by \( n^2 \left(\frac{2\sigma+1}{\tau}\right) \leq n^2 4^\sigma \) to get

\[
\leq b n^2 \left(\frac{8e^8 \sigma}{N}\right)^{(\sigma-1)/2} \left(\frac{16e^8 \sigma}{\tau}\right)^{(\tau-1)/2}
\]

So,

\[
\sum_{m=N-4\sigma}^{N-\sigma^{1/3}} \Pr(\exists \sigma, \tau, T_1, \ldots, T_\tau : M_1 + \cdots + M_\tau = m \mid M_1 + \cdots + M_\sigma = N) \\
\leq b n^2 N^2 \sigma \left(\frac{8e^8 \sigma}{N}\right)^{(\sigma-1)/2} \left(\frac{16e^8 \sigma}{\tau}\right)^{(\tau-1)/2} \\
= O(n^{-\text{any constant}})
\]

since \( \sigma \log N \gg \tau \log \sigma \).
When \( m \geq N - \sigma^{1/3} \) we replace (2) by

\[
\leq b \left( \frac{1 + \frac{\tau - 1}{2m}}{\frac{2m}{\tau - 1}} \right)^m \left( \frac{1 + \frac{2m}{\tau - 1}}{\frac{2N}{\sigma}} \right)^{(\tau - 1)/2} \sigma^{N - m} N^{1/2} \]

\[
\leq b \frac{e^{\tau/2 + o(\tau)} (\frac{2m}{\tau})^{(\tau - 1)/2} \sigma^{N - m} N^{1/2}}{e^{\sigma} (\frac{2N}{\sigma})^{(\sigma - 1)/2} m^{1/2}} \]

\[
\leq b \left( \frac{e^{1 + o(1)}}{\tau} \right)^{(\tau - 1)/2} \left( \frac{\sigma}{2N} \right)^{(\sigma - \tau)/2} \sigma^{\sigma/3}. \]

Inflating this by \( n^2 4^\sigma \) gives a bound of

\[
\leq b n^2 \left( \frac{16 e^{1 + o(1)}}{\tau} \right)^{(\tau - 1)/2} \left( \frac{8 \sigma^{1 + o(1)}}{N} \right)^{(\sigma - \tau)/2} = O(n^{-\text{any constant}}). \]

5 Modifications of Theorem 1

Let \( \lambda = \log^3 n \) and partition \( [\lambda] \) into \( q = \log n \) sets of size \( \lambda_1 = \log^2 n \). Now add \( n - \lambda \) vertices to \( T_\lambda \) and let \( M_i \) denote the number of vertices that land in the \( i \)th part \( \Pi_i \) of the partition. Lemma 2 implies that q.s.

\[
M_i \leq M_{\max} = \frac{200 n}{\log n} \log \log n, \quad 1 \leq i \leq \tau. \tag{4} \]

Let

\[
\omega_1(x) = \log^{\alpha/2} x \tag{5} \]

for \( x \in \mathbb{R} \).

Let \( L_i \) denote the length of the longest path in \( \Pi_i \). Suppose that \( T_n \) contains a path of length at least \( n/\omega_1, \omega_1 = \omega_1(n) \) and let \( k \) be the number of \( i \) such that

\[
L_i \geq \frac{200 n \log \log n}{\omega_1^2 \log n} \geq \frac{M_{\max}}{\log^\alpha(M_{\max})}. \]

Then, as \( k \leq q = \log n \) we have

\[
k \frac{200 n \log \log n}{\log n} + (\log n - k) \frac{200 n \log \log n}{\omega_1^2 \log n} \geq \frac{n}{\omega_1} \]
which implies that
\[ k \geq \frac{\log n}{201 \omega_1 \log \log n}. \]

Theorem 1 with the bound on \( M_i \) given in (4) implies that the probability of this is at most
\[ \frac{1}{n} + \left( \frac{\log n}{\log \alpha (n/ \log n)} \right)^{\log n} \frac{\log n}{201 \omega_1 \log \log n} \leq \frac{1}{n} + \left( \frac{1}{\log^{\alpha/3} n} \right)^{\log n} \frac{\log n}{201 \omega_1 \log \log n} \leq \frac{1}{\phi(n, \omega)} \]
where
\[ \phi(x, y) = \exp \left\{ \frac{\log x}{y \log \log x} \right\}. \]

The term \( 1/n \) accounts for the failure of the property in Lemma 2.

In summary, we have proved the following

**Lemma 3**
\[ \Pr \left( L(n) \geq \frac{n}{\omega_1(n)} \right) \leq \frac{1}{\phi(n, \omega)}. \] (7)

We are using \( \phi(x, y) \) in place of \( \phi(x) \) because we will need to use \( \omega_1(x) \) for values of \( x \) other than \( n \).

Next consider \( A_\sigma \) and \( \lambda_1 \leq \tau \ll \sigma \) and let \( T_1, T_2, \ldots, T_\tau \) be a set of \( \tau \) triangular faces of \( A_\sigma \). Suppose that we add \( N \gg \sigma \) more vertices and let \( N_j \) be the number of vertices that are placed in \( T_j \), \( 1 \leq j \leq \tau \).

Next let
\[ \Lambda(x) = e^{x^2} \] (8)
where \( x \in \mathbb{R} \).

Now let
\[ J = \{ j : N_j \geq \Lambda_0 \} \text{ where } \Lambda_0 = \Lambda(\omega_1(n)). \] (9)

Let \( L_j \) denote the length of the longest path through the ApN defined by \( T_j \) and the \( N_j \) vertices it contains, \( 1 \leq j \leq \tau \). For the remainder of the section let
\[ \omega_0 = \omega_1(\Lambda_0), \quad \phi_0 = \phi(\Lambda_0, \omega_0) = \exp \left\{ \frac{\omega_0}{2 \log \omega_0} \right\}, \quad \omega_2 = \frac{\phi_0}{\omega_0}. \] (10)

Then let
\[ J_1 = \left\{ j \in J : L_j \geq \frac{N_j}{\omega_1(N_j)} \right\}. \] (11)
We note that
\[
\log \omega_2 = \log \phi_0 - \log \omega_0 = \frac{\log \Lambda_0}{\omega_0 \log \log \Lambda} - \log \omega_0
\]
\[= \frac{\omega_0^2}{(2 + o(1))\omega_0 \log \log \omega_0} - \log \omega_0.
\]
For \(j \in J, N_j \geq \Lambda_0\) (see (9)). It follows from Lemma 3 that the size of \(J_1\) is stochastically dominated by \(Bin(\tau, 1/\phi_0)\). Using a Chernoff bound we find that
\[
\Pr \left( |J_1| \geq \frac{\omega_2 \tau}{\phi_0} \right) \leq \left( \frac{e}{\omega_2} \right)^{\omega_2 \tau/\phi_0}.
\]
(12)

Using this we prove

**Lemma 4** Suppose that
\[
\log \left( \frac{\sigma}{\tau} \right) \leq \frac{\omega_0}{\log \omega_0}.
\]
Then q.s., for all \(\lambda_1 \leq \tau \ll \sigma \ll N\) and all collections \(\mathcal{T}\) of \(\tau\) faces of \(\mathcal{A}_\sigma\) we find that with \(J_1\) as defined in (11),
\[
|J_1| \leq \frac{\omega_2 \tau}{\phi_0}.
\]

**Proof** It follows from (12) that
\[
\Pr \left( \exists \tau, \sigma, N, \mathcal{T} : |J_1| \geq \frac{\omega_2}{\tau \phi_0} \right)
\leq n^3 \left( \frac{2\sigma + 1}{\tau} \right) \left( \frac{e}{\omega_2} \right)^{\omega_2 \tau/\phi_0}
\leq n^3 \left( \frac{e(2\sigma + 1)}{\tau} \cdot \frac{e}{\omega_2} \right)^{\omega_2/\phi_0} \tau
\leq \exp \left\{ \tau \left( \frac{3 \log n}{\tau} + 2 + \log \left( \frac{\sigma}{\tau} \right) + \frac{\omega_2}{\phi_0} - \frac{\omega_2 \log \omega_2}{\phi_0} \right) \right\}
\leq \exp \left\{ \tau \left( \frac{3 \log n}{\tau} + 2 + \frac{\omega_0}{\log \omega_0} - \frac{\omega_0}{(2 + o(1)) \log \log \omega_0} \right) \right\}
= O(n^{-\text{any constant}}).
\]
6 Proof of Theorem 2

Fix a path $P$ of $A_n$. Suppose that after adding $\sigma \geq n^{1/2}$ vertices we find that $P$ visits

$$n^{1/2} \geq \tau \geq \lambda_1 \omega_0 \quad (13)$$

of the triangles $T_1, T_2, \ldots, T_\tau$ of $A_\sigma$. Now consider adding $N$ more vertices, where the value of $N$ is given in (16) below. Let $\sigma' = \sigma + N$ and let $\tau'$ be the number of triangles of $A_{\sigma'}$ that are visited by $P$.

We assume that

$$\frac{\alpha}{2} \log \log n \leq \log \left( \frac{\sigma}{\tau} \right) \leq \frac{\omega_0}{\log \omega_0}. \quad (14)$$

Let $M_i$ be the number of vertices placed in $T_i$ and let $N_i$ be the number of these that are visited by $P$. It follows from Lemma 2 that w.h.p.

$$\sum_{i=1}^\tau M_i \leq \frac{100 \tau \sigma}{\sigma} \log \left( \frac{\sigma}{\tau} \right).$$

Now w.h.p.,

$$\sum_{i=1}^\tau N_i \leq \tau \Lambda_0 + \frac{100 \omega_2 \tau N}{\phi_0 \sigma} \log \left( \frac{\sigma \phi_0}{\omega_2 \tau} \right) + \frac{100 \tau N}{\sigma \omega} \log \left( \frac{\sigma}{\tau} \right). \quad (15)$$

**Explanation:** $\tau \Lambda_0$ bounds the contribution from $[\tau] \setminus J$ (see (9)). The second term bounds the contribution from $J_1$. Now $|J_1| < \omega_2 \tau / \phi_0 \ll \tau$ as shown in Lemma 4. We cannot apply Lemma 2 to bound the contribution of $J_1$ unless we know that $|J_1| \geq \lambda_1$. We choose an arbitrary set of indices $J_2 \subseteq [\tau] \setminus J_1$ of size $\omega_2 \tau / \phi_0 - |J_1|$ and then the middle term bounds the contribution of $J_1 \cup J_2$. Note that $\omega_2 \tau / \phi_0 = \tau / \omega_0 \geq \lambda_1$ from (13). The third term bounds the contribution from $J \setminus J_1$. Here we use $\omega_1(N_j) \geq \omega_1(\Lambda_0) = \omega_0$, see (11).

We now choose

$$N = 3 \sigma \Lambda_0. \quad (16)$$

We observe that

$$\frac{\omega_2}{\phi_0} \log \left( \frac{\sigma \phi_0}{\omega_2 \tau} \right) \leq \frac{1}{\omega_0} \left( \frac{\omega_0}{\log \omega_0} + 2 \log \omega_0 \right) = o(1).$$

$$\frac{1}{\omega_0} \log \left( \frac{\sigma}{\tau} \right) \leq \frac{1}{\log \omega_0} = o(1).$$

Now along with Lemma 1 this implies that

$$\tau' \leq \sum_{i=1}^\tau (N_i + 1) \leq \tau + \tau \Lambda_0 + o \left( \frac{\tau N}{\sigma} \right).$$

Since $\sigma' = \sigma + N$ this implies that

$$\frac{\tau'}{\sigma'} \leq \left( \frac{1}{3} + o(1) \right) \frac{\tau}{\sigma} \leq \frac{\tau}{2 \sigma}.$$

It follows by repeated application of this argument that we can replace Theorem 1 by
Lemma 5

\[ \Pr \left( L(n) \geq \log n + \frac{100 \log n}{e^{\omega_0/\log \omega_0} n} \right) = O \left( \frac{1}{\phi(n, \omega_1(n))} \right). \]

**Proof** We add the vertices in rounds of size \( \sigma_0 = n^{1/2}, \sigma_1, \ldots, \sigma_m \). Here \( \sigma_i = 3\sigma_{i-1}\Lambda_0 \) and \( m - 1 \geq (1 - o(1)) \left( \frac{\log n}{\log \Lambda_0} \right) = (1 - o(1)) \left( \frac{\log n}{\omega_0(n)} \right)^2 = \log^{1-2\alpha} n \). We let \( P_0, P_1, P_2, \ldots, P_m = P \) be a sequence of paths where \( P_i \) is a path in \( A_i = A_{\sigma_0 + \cdots + \sigma_i} \). Furthermore, \( P_i \) is obtained from \( P_{i+1} \) in the same way that \( Q \) is obtained from \( P \) in Lemma 1. We let \( \tau_i \) denote the number of faces of \( A_i \) whose interior is visited by \( P_i \). It follows from Lemma 1 and Lemma 2 that the length of \( P \) is bounded by

\[ m + \frac{\tau_m - 1}{\sigma_m} \sigma_m \log \left( \frac{\sigma_m - 1}{\tau_m - 1} \right), \]

since the second term is a bound on the number of points in the interior of triangles of \( A_{m-1} \) visited by \( P \).

We have w.h.p. that

\[ \frac{\sigma_i}{\tau_i} \geq \begin{cases} 2\sigma_{i-1} \sigma_{i-1} & \frac{\sigma_{i-1}}{\tau_{i-1}} \leq e^{\omega_0/\log \omega_0} \\
100\tau_{i-1} \log(\sigma_{i-1}/\tau_{i-1}) & \frac{\sigma_{i-1}}{\tau_{i-1}} > e^{\omega_0/\log \omega_0}. \end{cases} \]

The second inequality here is from Lemma 2.

The result follows from \( 2^{\log^{1-2\alpha} n} \geq e^{\omega_0/\log \omega_0} \).

To get Theorem 2 we repeat the argument in Sections 5 and 6, but we start with \( \omega_1(x) = \log^{1/3} x \). The claim in Theorem 2 is then slightly weaker than the claim in Lemma 5.

**References**


