HAMILTON CYCLES IN RANDOM LIFTS OF DIRECTED GRAPHS

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Abstract

An *n-lift* of a digraph K, is a digraph with vertex set $V(K) \times [n]$ and for each directed edge $(i, j) \in E(K)$ there is a perfect matching between *fibers* $\{i\} \times [n]$ and $\{j\} \times [n]$, with edges directed from fiber i to fiber j. If these matchings are chosen independently and uniformly at random then we say that we have a random n-lift. We show that if h is sufficiently large then a random n-lift of the complete digraph $\vec{K}_h$ is hamiltonian **whp**.

1 Introduction

For a graph K, an *n-lift* G of K has vertex set $V(K) \times [n]$ where for each vertex $v \in V(K)$, $\{v\} \times [n]$ is called the *fiber* above v and will be denoted by F_v. The edge set of a an n-lift G consists of a perfect matching between fibers F_u and F_w for each edge $(u, w) \in E(K)$. The set of n-lifts will be denoted $\Lambda_n(K)$. In this paper we discuss random n-lifts, chosen uniformly from $\Lambda_n(K)$. In this case, the matchings between fibers are chosen independently and uniformly at random.

Lifts of graphs were introduced by Amit and Linial in [1] where they proved that if K is a connected, simple graph with minimum degree $\delta \geq 3$, and G is chosen randomly from $\Lambda_n(K)$ then G is $\delta-$connected **whp**, where the asymptotics are for $n \to \infty$. They continued the study of random lifts in [2] where they proved expansion properties of lifts. Together with Matoušek, they gave bounds on the independence number and chromatic number of random lifts in [3]. Linial and Rozenman [4] give a tight analysis for when a random n-lift has a perfect matching.

Burgin, Chebolu, Cooper and Frieze [6] showed that a random n-lift of the complete graph K_h is hamiltonian, provided h is sufficiently large. In this paper we study a directed version of the question. An *n-lift* of a digraph K, is a digraph with vertex set $V(K) \times [n]$ and for each directed edge $(i, j) \in E(K)$ there is a perfect matching between *fibers* $\{i\} \times [n]$ and $\{j\} \times [n]$, with edges directed from fiber i to fiber j.

We use the notation $y \overset{r}{\in} Y$ for "y is chosen uniformly at random from Y". We let $\vec{K}_h$ denote the complete digraph on vertex set $[h]$. Note that here there are edges in both directions (u, v)

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and (v,u) for all $u \neq v \in [h]$.

Theorem 1. *If h is sufficiently large and $D \overset{r}{\in} \Lambda_n(\vec{K}_h)$ then D is hamiltonian* **whp**.

We will use the 3-phase method used in Cooper and Frieze [7, 8], Cooper, Frieze and Molloy [9] and Frieze, Karp and Reed [12].

A *permutation digraph* is a set of vertex disjoint directed cycles that cover all n vertices. Its *size* is the number of cycles.

Phase 1. We show that **whp** the lift D contains a directed permutation digraph of size at most $2\ln n$.

Phase 2. We increase the minimum cycle length in the permutation digraph to at least

$$n_0 = \left\lceil \frac{100nh^3}{\ln n} \right\rceil.$$

Phase 3. We convert the *Phase 2* permutation digraph to a Hamilton cycle.

The main difficulty involved in implementing this strategy comes from Phase 3. This is basically a second moment calculation, but it needs a *trick* to reduce the variance. The idea of the trick is from [7], but implementing the idea has turned out to be quite difficult. This is basically the content of Section 4 where we prove a lower bound on the number of Hamilton cycles of a certain type in a digraph of high degree.

We use the following standard inequalities for the tails of the binomial distribution:

$$\mathbf{Pr}(|B(n,p) - np| \geq \epsilon np) \leq 2e^{-\epsilon^2 np/3}, \qquad 0 \leq \epsilon \leq 1, \tag{1}$$

$$\mathbf{Pr}(B(n,p) \geq anp) \leq (e/a)^{anp}. \tag{2}$$

2 Phase 1. Making a permutation digraph with at most $2\log n$ cycles

Lemma 1. *Suppose that $D \overset{r}{\in} \Lambda_n(\vec{K}_h)$. Then* **whp** *$D$ contains a permutation digraph with at most $2\ln n$ cycles.*

Proof Let X_0 denote the Hamilton cycle $(1,2,...,h)$ of K_h. Let F_i be the fiber of D corresponding to i. Let $r_{i,j}$ be the permutation defined by the matching $M_{i,j}$ *from* fiber F_i to fiber F_j in the lift D i.e. the edges of $M_{i,j}$ are $\{(i,k),(j,r_{i,j}(k)) : k \in [n]\}$. Let $r = r_{h,1} \circ r_{h-1,h} \circ \cdots \circ r_{1,2}$. r defines a permutation of fiber F_1. The permutation digraph $\{(i,x),(i+1,r_{i,i+1}(x)) : i \in [h], x \in [n]\}$ has as many cycles as the permutation r. The permutation r is a random permutation as it is the composition of random permutations. We know that the number of cycles in a random permutation is at most $2\ln n$ **whp**, see for example Bollobás [5]. □

We partition the cycles of the permutation digraph Σ_0 into sets SMALL and LARGE, containing small cycles C of length $|C| < n_0$ and large cycles $|C| \geq n_0$ respectively.

In a random permutation the expected number of vertices on cycles of length at most n_0 is precisely n_0 ([13]). Thus, by the Markov inequality, **whp** Σ_0 contains at most $nh\log\log n/(4\log n)$ vertices on small cycles.

Thus at the end of Phase 1, we can assume we have a permutation digraph Σ_0 of size at most $2\ln n$ and which contains at most $nh\log\log n/(4\log n)$ vertices on cycles of length $\leq n_0$. Let E_0 denote the edges D that are **not** in Σ_0.

3 Phase 2. Removing small cycles

We now denote the vertices in the lift by $v_{i,k}$ where $i \in [h]$ and $k \in [n]$. We define a Near Permutation Digraph (NPD) to be a digraph obtained from a permutation digraph by removing one edge. Thus an NPD Γ consists of a path $P(\Gamma)$ plus a permutation digraph $PD(\Gamma)$ which covers $([h] \times [n]) \setminus V(P(\Gamma))$.

Each step of the process we are about to describe involves the exposure of an edge $(v_{i,k}, v_{i',k'})$. When an edge is exposed in this way in Phase 2, we say that the two endpoints are *used.* The set of used vertices is denoted by W. Initially, $W = \emptyset$, and we ensure that $|W| \leq n^{3/4}$ throughout. At any time therefore, the process is conditioned by the knowledge of partial matchings $M'_{i,j}$ between the fibers F_i and F_j for $i \neq j$. (The matchings $M_{i,i+1}$ have of course been completely exposed in Phase 1). For $j \neq i, i+1$, the unexposed part of $M_{i,j}$ will be a uniform extension of $M'_{i,j}$. So, in particular, when we examine a vertex $v_{i,k} \notin W$, the $M_{i,j}$ edge incident with $v_{i,k}$ is chosen uniformly from a set of size $n - o(n)$.

We now give an informal description of a process which removes a small cycle C from a *current* permutation digraph Σ. We break this process into an *Out-Phase* and an *In-Phase.* We start by choosing an (arbitrary) edge (v_{i,j_0}, v_{i+1,k_0}) of C and delete it to obtain an NPD Γ_0 with $P_0 = P(\Gamma_0) \in \mathcal{P}(v_{i+1,k_0}, v_{i,j_0})$, where $\mathcal{P}(x, y)$ denotes the set of paths from x to y in D. The aim of the process is to produce a *large* set S of NPD's such that for each $\Gamma \in S$, (i) $P(\Gamma)$ has a least n_0 edges and (ii) the small cycles of $PD(\Gamma)$ are a strict subset of the small cycles of Σ. We will show that **whp** the endpoints of one of the $P(\Gamma)$'s can be joined by an edge to create a permutation digraph with (at least) one less small cycle. This completes the informal description.

We now give a fairly formal description, but we leave out some details for later. We produce a sequence $S_0 = \{\Gamma_0\}, S_1, S_2, \ldots, S_t, \ldots$ of sets of NPD's. Fix $t > 0$ and $\Gamma \in S_{t-1}$ and let v be the terminal endpoint of $P(\Gamma)$. We examine the $h-2$ edges of E_0 leaving v i.e. the edges going *out* from the end of the path. Let w_j, $1 \leq j \leq h-2$ be the terminal vertices of these edges, and assume that Γ contains edges (x_j, w_j), $1 \leq j \leq h-2$. Then for $1 \leq j \leq h-2$, $\Gamma_j = \Gamma \cup \{(v, w_j)\} \setminus \{(x_j, w_j)\}$ is added to S_t, assuming that the edge (v, w_j) is *acceptable* w.r.t. Γ. We call this an acceptable *out-step.* An (v, w) is acceptable if the following is true: Suppose that $P(\Gamma) \in \mathcal{P}(., v)$. Let $\Gamma' = \Gamma \cup \{(v, w)\} \setminus \{(x, w)\}$ where $(x, w) \in E(\Gamma)$. We say that we *use* w.

(i) $P(\Gamma')$ contains at least n_0 edges.

(ii) Any new cycle created (i.e. in Γ' and not in Γ) also has at least n_0 edges.

(iii) w does not lie on a small cycle of Γ.

(iv) $P(\Gamma') \in \mathcal{P}(., x)$ where x has not been used before in Phase 2.

If Γ_j contains no edge (x_j, w_j) then $w_j = v_{i+1,k_0}$. We accept the edge if $P(\Gamma_j)$ has at least n_0 edges. This would create a PD and (prematurely) end an iteration, although it is unlikely to occur.

Each member of S_{t-1} (usually) has $h-2$ descendants in S_t and in this way we build a tree T_0 of NPD's in a natural breadth-first fashion where each non-leaf Γ at depth t is an element of S_t. By construction, all paths $P(\Gamma)$, $\Gamma \in T_0$ will have the same start vertex viz. the head v_{i+1,k_0} of the edge deleted from the small cycle C. The construction of T_0 ends when we first have $\nu = \left\lceil \sqrt{n \log n} \right\rceil$ leaves. The construction of T_0 constitutes an Out-Phase of our procedure to eliminate small cycles. Having constructed T_0 we need to do a further *In-Phase*, which is similar to a set of Out-Phases.

Then **whp** we close at least one of the paths $P(\Gamma)$ to a cycle of length at least n_0. If this process fails then we try again with a different independent edge of C in place of (v_{i,j_0}, v_{i+1,k_0}). If we succeed we move on to the next small cycle.

We now fill in the details. We start Phase 2 with a permutation digraph Σ_0 and a general iteration of Phase 2 starts with a permutation digraph Σ whose small cycles are a subset of those in Σ_0. Iterations continue until there are no more small cycles. At the start of an iteration we choose some small cycle C of Σ. There then follows an Out-Phase in which we construct a tree $T_0 = T_0(\Sigma, C)$ of NPD's as follows: The root of T_0 is Γ_0 which is obtained by deleting an edge (v_{i,j_0}, v_{i+1,k_0}) of C.

We grow T_0 to a depth $O(\log n)$. The set of nodes at depth t will be S_t. $\Gamma \in S_{t-1}$ with $P = P(\Gamma) \in \mathcal{P}(v_{i,j_0}, v)$, $v \in F_i$, has up to $h-2$ descendants in S_t.

Lemma 2. *Let $C \in SMALL$. Then, where $\nu = \lceil \sqrt{n \log n} \rceil$,*

$$\mathbf{Pr}(\exists t < \left\lceil \log_{h/2} \nu + 500 \log\log n \right\rceil \text{ such that } |S_t| \in [\nu, h\nu]) = 1 - O((\log\log n)^3/\log n).$$

Proof We assume that we stop construction of T_0, in mid-phase if necessary, when $|S_t| \in [\nu, h\nu]$. Let us consider a generic construction in the growth of T_0.

For an NPD Γ with $P(\Gamma) \in \mathcal{P}(v_{i,j_0}, v)$ we let $Z_j(\Gamma)$ be the 0-1 indicator for the edge (v, w_j) being unacceptable. If $Z_j(v) = 1$ then either (a) w_j lies on $P(\Gamma)$ and is too close to an endpoint; this has probability bounded above by $201h^3/\log n$, or (b) the corresponding vertex x_j is in W; this has probability bounded above by $2n^{-1/4}$, or (c) w_j lies on a small cycle of Γ and hence of Σ_0; this has probability bounded above by $\log\log n/3\log n$. Then $\mathbf{Pr}(Z_j(\Gamma) = 1) \le \log\log n/2\log n$ regardless of the history of the process to this point.

Let $Z_t = \sum_{\Gamma \in S_{t-1}} \sum_j Z_j(\Gamma)$. Z_t is the sum of possibly dependent random variables, but it is stochastically dominated by the binomial $B((h-2)|S_t|, \log\log n/\log n)$.

We write

$$|S_{t+1}| = (h-2)|S_t| - Z_t.$$

Now let $t_0 = \lceil 1000 \log\log n \rceil$, $t_1 = \left\lceil \log_{h/2} \nu + 1000 \log\log n \right\rceil$.

(a) $\mathbf{Pr}(\exists t \le t_0 : |S_t| \le 500 \log\log n \text{ and } Z_t > 0) = O((\log\log n)^3/\log n)$

(b) $\mathbf{Pr}(\exists t \le t_1 : |S_t| \ge 500 \log\log n \text{ and } Z_t > h|S_t|/100) \le (\log n)^{-\Omega(\log\log n)}$.

(a) $\mathbf{Pr}(Z_t > 0 \mid\ |S_t| \le 500 \log\log n) = O((\log\log n)^2/\log n)$ by the Markov inequality.

(b) Immediate from (2).

Let $\mathcal{E}_a$ and $\mathcal{E}_b$ be the low probability events described in (a) and (b) above. Assume the occurrence of $\bar{\mathcal{E}}_a \cap \bar{\mathcal{E}}_b$. $\bar{\mathcal{E}}_a$ implies that $|S_t|$ reaches size at least $500 \log\log n$ before t reaches t_0. Once this happens, $\bar{\mathcal{E}}_b$ implies that $|S_t|$ then grows geometrically with t up to time t_1 at a rate of at least h/2. The lemma follows. □

The total number of vertices added to W in this way throughout the whole of Phase 2 is $O(\nu|SMALL|) = o(n^{3/4})$. We try this process once or twice for each $C \in SMALL$. Let t^* denote the value of t when we stop the growth of T_0. At this stage we have leaves Γ_k, for $k = 1, \ldots, \nu$, each with a path of length at least n_0, (unless we have already successfully made a cycle).

We now execute an In-Phase. This involves the construction of trees $T_k, k = 1, 2, \ldots \nu$. The reader may while reading become concerned that in building ν trees of size ν we will need to use too many vertices. This will not be the case. This is a consequence of (3) below. Assume that $P(\Gamma_k) \in \mathcal{P}(v_{i+1,k_0}, v_{i,\ell_k})$. Notice that the start vertex of each of these paths is the same viz. the head v_{i+1,k_0} of the edge deleted from the small cycle C. We start with Γ_k and build T_k in a similar way to T_0 except that here all paths generated end with v_{i,ℓ_k}. This is done as follows: If a current NPD Γ has $P(\Gamma) \in \mathcal{P}(u, v_{i,\ell_k})$ then we consider adding an edge $(w, u) \in E_0$ and deleting an edge $(w, x) \in \Gamma$. Thus our trees are grown by considering edges directed into the start vertex of each $P(\Gamma)$ rather than directed out of the end vertex. Some technical changes are necessary however. We consider the construction of our ν trees in two stages. First of all we grow the trees only enforcing (an in-analogue of) condition (iv) of acceptability and thus allow the formation of small cycles and paths. We try to grow them to depth t_1. The growth of the ν trees can naturally be considered to occur simultaneously. Let $L_{k,\ell}$ denote the set of start vertices of the paths associated with the nodes at depth ℓ of the kth tree, $k = 1, 2 \ldots, \nu, \ell = 0, 1, \ldots, t_1$. Thus $L_{k,0} = \{v_{i+1,k_0}\}$ for all k. We prove inductively that

$$L_{k,\ell} = L_{1,\ell} \ for\ all\ k, \ell. \tag{3}$$

In fact if $L_{k,\ell} = L_{1,\ell}$ then the acceptable E_0 edges have the same set of initial vertices and since all of the deleted edges are Σ_0-edges (enforced by (iv)) we have $L_{k,\ell+1} = L_{1,\ell+1}$. Note that the number of nodes in each tree is $O(h^{t_1+1}) = O(n^{3/5})$ if h is sufficiently large. Although we grow many trees, because of (3), the actual number of vertices used in total is $O(n^{3/5})$ and this is why we can claim that **whp** $|W| \leq n^{3/4}$ throughout.

The probability that we succeed in constructing trees $T_1, T_2, \ldots T_\nu$ is, by the analysis of Lemma 3, $1 - O((\log \log n)^3/\log n)$.

We now consider the fact that in some of the trees some of the leaves may have been constructed in violation of (i)–(iii). We imagine that we prune the trees $T_1, T_2, \ldots T_\nu$ by disallowing any node that was constructed in violation of (i)–(iii). Let a tree be BAD if after pruning it has less than ν leaves and GOOD otherwise. Now an individual pruned tree has been constructed in the same manner as the tree T_0 obtained in the Out-Phase. (We have chosen t_1 to obtain ν leaves even at the slowest growth rate of h/2 per node as asked for at the end of the proof of Lemma 2.) Thus

$$\mathbf{Pr}(T_1 \text{ is BAD}) = O\left(\frac{(\log \log n)^3}{\log n}\right)$$

and

$$\mathbf{E}(\text{number of BAD trees}) = O\left(\frac{\nu(\log \log n)^3}{\log n}\right)$$

and

$$\mathbf{Pr}(\exists \geq \nu/2 \text{ BAD trees}) = O\left(\frac{(\log \log n)^3}{\log n}\right).$$

Thus

$$\begin{aligned}
&\mathbf{Pr}(\exists < \nu/2 \text{ GOOD trees after pruning}) \\
&\qquad \leq \mathbf{Pr}(\text{failure to construct } T_1, T_2, \ldots T_\nu) + \mathbf{Pr}(\exists \geq \nu/2 \text{ BAD trees}) \\
&\qquad = O\left(\frac{(\log \log n)^3}{\log n}\right)
\end{aligned}$$

Thus with probability 1-$O((\log \log n)^3/\log n)$ we end up with $\nu/2$ sets of ν paths, each of length at least $100nh^3/\log n$. All paths in one set have the same terminal vertex. Suppose that $v \in F_i$

is the common terminal vertex of one set. If a path in this set begins with a vertex in F_{i+1} then we attempt to change the fiber of its start vertex to F_k, $k \neq i+1$ by performing an acceptable in-step. We will succeed with probability $1-o(1)$, given the previous history. Thus assume that we have $\nu/2$ sets of $\nu/2$ paths where if the initial vertex is in fiber F_i then the terminal vertex is not in fiber F_{i-1}. Given this,

$$\begin{aligned}\mathbf{Pr}(\text{no } E_0 \text{ edge closes one of these paths}) &\leq \left(1-\frac{\nu}{2n(1-o(1))}\right)^{\nu/2} \\ &= O(n^{-1/4}).\end{aligned}$$

Consequently the probability that we fail to eliminate a particular small cycle C after breaking an edge is $O((\log\log n)^3/\log n)$. For every $C \in SMALL$ we have $|C| \geq h$ and so it is possible to try once or twice using independent edges of C and so the probability that we fail to eliminate a given small cycle C is certainly $O((\log\log n)^6/(\log n)^2)$ (remember that we calculated all probabilities conditional on previous outcomes and assuming $|W| \leq n^{3/4}$.) Hence, since **whp** $|C| = O(\log n)$,

Lemma 3. *The probability that Phase 2 fails to produce a permutation digraph with minimal cycle length at least n_0 is $o(1)$.*

At this stage we have shown that if h is sufficiently large than D almost always contains a permutation digraph Σ^* in which the minimum cycle length is at least n_0.

We shall refer to Σ^* as the *Phase 2* permutation digraph.

Now all the cycles of the PD Σ_0 defined in Phase 1 have lengths divisible by h. Also, as we traverse a cycle the fibers encountered are $F_1, F_2, \ldots, F_h, F_1, \ldots,$. Now Σ^* is obtained from Σ_0 by replacing $O((\log n)^2)$ edges. By considering the unbroken sections of Σ_0 which pass through the fibers in order, we see that for each i, a cycle of length ℓ in Σ^* contains at least $\ell/h - O((\log n)^2)$ edges from fiber F_i to F_{i+1}.

4 Deterministic Problem

In this section we give a lower bound to the number of a certain type of Hamilton cycle in a digraph with large minimum in-degree and out-degree.

Let Γ be a digraph with vertex set $[m]$ and minimum in-degree and out-degree at least $.99m$.

For a permutation ρ of $[m]$, let PD_ρ be its associated permutation digraph. Let

$$\begin{aligned}T_\Gamma &= \{\rho : \ PD_\rho \text{ is a sub-graph of } \Gamma\} \\ S_\Gamma &= \{\rho \in T_\Gamma : \ \rho \text{ is cyclic}\}.\end{aligned}$$

Let ϕ be a fixed *even* permutation of $[m]$. Let

$$R_\Gamma = \{\rho \in S_\Gamma : \ \phi\rho \text{ is cyclic}\}.$$

The permutations ρ thus correspond to a restricted class of Hamilton cycles in Γ.

In the next few sections we will prove that

Theorem 2. $|R_\Gamma| \geq m! e^{-3m}$.

This will help us resolve a strange technical problem, already met in [7]. It facilitates a second moment calculation.

The proof is quite long and it is deferred to a later section, so as not to interrupt the flow of the probabilistic part of the argument.

5 Second Moment Calculation

Let $C_1, C_2, \ldots, C_k$ be the cycles of Σ^* produced by Phase 2, and let $c_i^* = \min_j c_{ij}$ where c_{ij} is the number of *clean* edges in cycle C_i from fiber F_j to F_{j+1} for $j = 1, 2..., h$. An edge is clean if it is not incident with W. Recall that W is the set of vertices v for which Phase 2 exposed an edge incident with v. E_1 will denote the set of clean edges that do not join fibers F_i, F_{i+1} for some i.

The cycles are numbered so that $c_1^* \leq c_2^* \leq ... \leq c_k^*$ and $c_1^* \geq n_0/h - 2n^{3/4} \geq \frac{99h^2n}{\log n}$. If $k = 1$ there is nothing more to do. Otherwise let $a = \left\lceil \frac{nh^2}{\log n} \right\rceil$. We will show that **whp** it is possible to delete a set of edges from each C_i and then replace them so that the resulting structure is a Hamilton cycle. We will use the second moment method to do this.

We select an odd number of edges from each C_i and delete them. The choice of parity is related to the need to keep permutation ϕ (defined next) even. See (4). We will then be able to apply Theorem 2. Continuing, when h is odd, we choose the edges in the following manner. For each C_i, we select a set of $l_i = 2\lfloor \frac{c_i*}{a} \rfloor + 1$ vertices $v \in C_i \setminus W$ from each fiber F_j, where $j = 1, 2, ..., h$, and delete the corresponding edges (v, u) in Σ^*. The number of edges deleted from cycle C_i is $m_i = l_i h$, which is odd. Since $v \in C_i \setminus W$, the deleted edge $(v, u) \in \Sigma_0$ is an edge between fibers F_j and F_{j+1} for some j. The above procedure is not acceptable when h is even as we would end up deleting an even number of edges from each cycle. We circumvent this problem by choosing $l_i - 1$ vertices from fiber F_j where $j \equiv i \pmod h$ and l_i from the rest of the fibers for cycle C_i. This ensures that the number of edges deleted from cycle C_i is $m_i = l_i h - 1$, which is odd.

Let $m = \sum_{i=1}^{k} m_i$. In each cycle C_i choose a vertex x_i which loses a cycle edge directed out of it. Let $v_1 = x_1$ and then go round C_1 defining $v_2, v_3, \ldots v_{m_1}$ in order as the end points of the path sections. Then let $v_{m_1+1} = x_2$ and so on. Now re-label the broken edges as $(v_i, u_i), i \in [m]$. We thus have m path sections $P_j \in \mathcal{P}(u_{\phi(j)}, v_j)$ in Σ^* for permutation ϕ where $\phi(1) = m_1, \phi(2) = 1, \ldots \phi(m_1) = 1, \phi(m_1 + 1) = m_1 + m_2, \phi(m_1 + 2) = m_1 + 1$ etc.. (Some path sections could be just a single vertex).

Note that since ϕ is made up of cycles of odd length,

$$\phi \text{ is an even permutation of } [m]. \tag{4}$$

The number of paths starting (ending) at any of the fibers is either $\lfloor \frac{m}{h} \rfloor$ or $\lfloor \frac{m}{h} \rfloor + 1$.

We will attempt to re-join these paths in a different order so that a Hamilton cycle is constructed. Suppose that path section P starts in fiber $F_\xi(P)$ and ends in fiber $F_\eta(P)$. Because we do not wish to put back edges that we have just deleted: If P immediately precedes Q in some re-ordering then we will have $\xi(Q) \neq \eta(P), \eta(P) + 1$. We write $P \to Q$ if this holds for P, Q.

Fix for now, a choice of $P_1, P_2, \ldots, P_m$ and let Γ be the digraph with vertex set $[m]$ and a directed edge (i, j) whenever $P_i \to P_j$. We are interested in the number of hamilton cycles in Γ, that satisfy a certain property. Each vertex of Γ has in-degree and out-degree at least $m - 2\lfloor \frac{m}{h} \rfloor - 2$. By choosing h to be suitably large, we can assume that $m - 2\lfloor \frac{m}{h} \rfloor - 2 \geq .99m$. Thus, Γ satisfies the conditions of Theorem 2.

Let Ω denote the set of ordered pairs of selections of edges for deletion and and cycle re-arrangements ρ satisfying the condition that a path ending on fiber F_i is not joined to a path starting on the same fiber or F_{i+1} and such that $\lambda = \phi\rho$ is cyclic. $\omega \in \Omega$ is a *success* if E_1 contains the edges needed for the associated Hamilton cycle.

Let H stand for the union of the permutation digraph Σ^* and E_1. We finish our proof by proving

Lemma 4. $\mathbf{Pr}$*(*H *does not contain a Hamilton cycle)* $= o(1)$.

Proof.

Let X be the number of Hamilton cycles in H obtainable by deleting edges as above, re-arranging the path sections generated by ϕ and if possible reconnecting all the sections using edges of E_1. We can think of this as the sum of indicator variables indexed by Ω. We will use the well-known inequality

$$\mathbf{Pr}(X > 0) \geq \frac{\mathbf{E}(X)^2}{\mathbf{E}(X^2)}. \tag{5}$$

Probabilities in (5) are thus with respect to the space of E_1 choices for edges incident with vertices not in W.

Now the definition of l_i yields that

$$\frac{2nh - |W|h}{a} - (h+1)k \leq m \leq \frac{2nh}{a} + hk$$

and so

$$\frac{1.98}{h}\log n \leq m \leq \frac{2.01}{h}\log n.$$

Also

$$k \leq \log n/100h^3, \quad l_i \geq 199 \text{ and } \frac{c_i^*}{l_i} \geq \frac{a}{2.02}, \qquad 1 \leq i \leq k.$$

Now fix a set of m paths and a permutation $\omega \in \Omega$. The probability that the edges exist for a success is at least n^{-m}. For having conditioned on the existence of a set of edges A, the probability that edge (u, v) exists is $1/\mu$ where $\mu \leq n$ is the number of vertices in the fiber containing v which are not incident with an A-edge whose other endpoint is in the same fiber as u. Recall that up this point all we have done is to conditoin on certain edges being present. The remaining edges form random partial matchings between the fibers. Furthermore, we have only conditioned on the presence of $O(n^{3/4})$ edges in total.

Let $\theta = 1_{h \text{ is even}}$ and $\theta_{ij} = \theta \times 1_{i \equiv j \mod h}$.

$$\begin{aligned}
\mathbf{E}(X) &= \sum_{\omega\in\Omega} \mathbf{Pr}(\omega \text{ is a success}) \\
&\geq \sum_{\omega\in\Omega} n^{-m} \\
&\geq n^{-m} m! e^{-3m} \prod_{i=1}^{k} \binom{c_i^*}{l_i - \theta} \prod_{\substack{j=1\\ j\neq i}}^{h} \binom{c_i^*}{l_i} \\
&\geq n^{-m} m! e^{-3m} \prod_{i=1}^{k} \binom{c_i^*}{l_i}^h \left(\frac{l_i}{c_i^*}\right)^\theta \qquad \textit{after using Theorem 2} \\
&\geq e^{-3m} \left(\frac{m}{en}\right)^m \prod_{i=1}^{k} \left(\left(\frac{c_i^* e^{1-1/12l_i}}{l_i^{1+(1/2l_i)}} \right)^{l_i} \left(\frac{1-2l_i^2/c_i^*}{\sqrt{2\pi}} \right) \right)^h \left(\frac{l_i}{c_i^*}\right)^\theta \\
&\geq e^{-3m} \left(\frac{m}{en}\right)^m \prod_{i=1}^{k} \left(\frac{e}{1.01}\right)^{l_i h} \left(\frac{c_i^*}{l_i}\right)^{l_i h - \theta} \\
&= e^{-3m} \left(\frac{mea}{(1.01)(2.02)en} \right)^m \\
&\geq e^{-3m} (.9h)^m \\
&\geq n^{\Omega(1)}. \qquad (6)
\end{aligned}$$

Let M, M' be two sets of selected edges which have been deleted in Σ^* and whose path sections have been rearranged into Hamilton cycles according to ρ, ρ' respectively. Let N, N' be the corresponding sets of edges which have been added to make the Hamilton cycles. What is the interaction between these two Hamilton cycles?

Let $s = |M \cap M'|$ and $t = |N \cap N'|$. Now $t \leq s$ since if $(v, u) \in N \cap N'$ there must be a unique $(\tilde{v}, u) \in M \cap M'$ which is the unique Σ^*-edge into u.

We claim that $t = s > 0$ implies $t = s = m$ and $(M, \rho) = (M', \rho')$. (This is why we have restricted our attention to $\rho \in R_\Gamma$.) For the following argument recall that we delete edges (v_i, u_i), $i \in [m]$ and our path segments go from $u_{\phi(i)}$ to v_i and in our re-ordering, v_i will be connected to $u_{\phi\rho(i)} = u_{\lambda(p)}$. Suppose then that $t = s > 0$ and $(v_i, u_i) \in M \cap M'$. Now the edge $(v_i, u_{\lambda(i)}) \in N$ and since $t = s$ this edge must also be in N'. But this implies that $(v_{\lambda(i)}, u_{\lambda(i)}) \in M'$ and hence in $M \cap M'$. Repeating the argument we see that $(v_{\lambda^k(i)}, u_{\lambda^k(i)}) \in M \cap M'$ for all $k \geq 0$. But λ is cyclic and so our claim follows.

We adopt the following notation. Let $\langle s, t \rangle$ denote $|M \cap M'| = s$ and $|N \cap N'| = t$. So

$$\begin{aligned}
\mathbf{E}(X^2) &\leq \mathbf{E}(X) + (1 + O(m|W|/n)) \sum_{\omega\in\Omega} n^{-m} \sum_{\omega' : N' \cap N = \emptyset} n^{-m} \\
&\quad + (1 + O(m|W|/n)) \sum_{\omega\in\Omega} n^{-m} \sum_{s=2}^{m} \sum_{t=1}^{s-1} \sum_{\omega' : \langle s,t \rangle} n^{t-m} \\
&= \mathbf{E}(X) + (1 + o(1))(A_1 + A_0) \text{ say.} \qquad (7)
\end{aligned}$$

Clearly

$$A_1 \leq \left(\sum_{\omega\in\Omega} n^{-m} \right)^2 \leq \mathbf{E}(X)^2. \qquad (8)$$

For given M, M', ρ, how many ρ' satisfy the condition $\langle s,t\rangle$? We bound it from above by $(m-t)!$ (consider fixing t edges of Λ').
Thus

$$\begin{aligned} A_0 &\leq \sum_{\omega\in\Omega} n^{-m} \sum_{s=2}^{m}\sum_{t=1}^{s-1}\left[\sum_{\sigma_{11}+\cdots+\sigma_{kh}=s}\prod_{i=1}^{k}\prod_{j=1}^{h}\binom{l_i-\theta_{ij}}{\sigma_{ij}}\binom{c_i^*-l_i}{l_i-\sigma_{ij}}\right](m-t)!n^{t-m} \\ &\leq \mathbf{E}(X)^2\sum_{s=2}^{m}\sum_{t=1}^{s-1}\binom{s}{t}\left[\sum_{\sigma_{11}+\cdots+\sigma_{kh}=s}\prod_{i=1}^{k}\prod_{j=1}^{h}\frac{\binom{l_i-\theta_{ij}}{\sigma_{ij}}\binom{c_i^*-l_i}{l_i-\sigma_{ij}}}{\binom{c_i^*}{l_i}}\right]\frac{(m-t)!e^{3m}}{m!}n^t. \end{aligned}$$

Now

$$\frac{\binom{c_i^*-l_i}{l_i-\sigma_{ij}}}{\binom{c_i^*}{l_i}} \leq \frac{\binom{c_i^*}{l_i-\sigma_{ij}}}{\binom{c_i^*}{l_i}} \leq (1+o(1))\left(\frac{l_i}{c_i^*}\right)^{\sigma_{ij}} \leq (1+o(1))\left(\frac{2.02}{a}\right)^{\sigma_{ij}}$$

where the $o(1)$ term is $O((\log n)^3/n)$. Also

$$\sum_{\sigma_{11}+\cdots+\sigma_{kh}=s}\prod_{i=1}^{k}\prod_{j=1}^{h}\binom{l_i-\theta_{ij}}{\sigma_{ij}} = \binom{m}{s}.$$

Hence,

$$\begin{aligned} \frac{A_0}{\mathbf{E}(X)^2} &\leq (1+o(1))\sum_{s=2}^{m}\sum_{t=1}^{s-1}\binom{s}{t}\left(\frac{2.02}{a}\right)^s\binom{m}{s}\frac{(m-t)!e^{3m}n^t}{m!} \\ &\leq n^{.01}\sum_{s=2}^{m}\sum_{t=1}^{s-1}\binom{s}{t}\left(\frac{2.02}{a}\right)^s\frac{m^{s-t}n^t}{s!}, \qquad \textit{using } e^{3m}\leq n^{.005} \textit{ for } h \textit{ large}, \\ &= n^{.01}\sum_{s=2}^{m}\left(\frac{2.02}{a}\right)^s\frac{m^s}{s!}\sum_{t=1}^{s-1}\binom{s}{t}\left(\frac{n}{m}\right)^t \\ &\leq \frac{2m}{n^{.99}}\sum_{s=2}^{m}\left(\frac{(2.02)n}{a}\right)^s\frac{1}{s!} \\ &\leq \frac{2m}{n^{.99}}\sum_{s=2}^{m}\left(\frac{2.02\log n}{h^2}\right)^s\frac{1}{s!} \\ &\leq \frac{2m}{n^{.99}}n^{2.02/h^2} \\ &= o(1). \end{aligned} \tag{9}$$

The lemma follows from (5) to (9). □

This completes the proof of Theorem 1.

6 Proof of Theorem 2

6.1 Lower bound for $|T_\Gamma|$:

Lemma 5. $|T_\Gamma| \geq m!e^{-5m/2}$.

Proof Consider a bipartite graph $B = (V_1 \cup V_2, E)$ where V_1, V_2 are disjoint copies of $[m]$. The edge set $E(B) = \{(i,j) : (i,j) \in E(\Gamma)\}$. It follows from the above that the degree of each vertex is at least $\delta_B = .99m$. A perfect matching M in B gives rise to a member ρ of T_Γ where $j = \rho(i)$ iff $(i,j) \in M$. For a matching M of $K_{m,m}$, let $\nu(M)$ be the number of edges of M which are also in B. Let $\mathcal{M}_k$ denote the set of perfect matchings of $K_{m,m}$ with $\nu(M) = k, k = 0, 1, 2, ..., m$. We first prove that if $\mu_k = |\mathcal{M}_k|, k = 0, 1, ..., m$ and $k \geq .26m + 1$, then

$$\frac{\mu_{k+1}}{\mu_k} \geq \frac{.23m(m-k)}{\binom{m}{2}} \geq \frac{.46(m-k)}{m}. \tag{10}$$

Consider the set $\mathcal{X}$ of ordered pairs (M_1, M_2) where $M_1 \in \mathcal{M}_k$ and $M_2 \in \mathcal{M}_{k+1}$ and the symmetric difference $M_1 \oplus M_2$ is an alternating cycle of length 4. Now each $M \in \mathcal{M}_k$ is in at least

$$(m-k)(2(\delta_B - 1) - (m-1) - (m-k)) \geq .23m(m-k)$$

such cycles.

Explanation: There are $m - k$ choices for an edge $e = (v, w) \in M \setminus E(B)$. Then there are at least $2(\delta_B - 1) - (m-1)$ pairs of vertices (a, b) such that $(a, b) \in M$ and $(v, b) \in E(B)$, $(a, w) \in E(B)$ and at least $2(\delta_B - 1) - (m-1) - (m-k)$ pairs $(a, b) \in M \cap E(B)$, in which case $M' = (M \cup (a, w), (v, b)) \setminus ((v, w), (a, b))$ is a member of $\mathcal{M}_{k+1}$ and $(M, M') \in \mathcal{X}$.
On the other hand, each M is in at most $\binom{m}{2}$ pairs and (10) follows.

Now, if $k \leq .26m$, then

$$\frac{\mu_{k+2}}{\mu_k} \geq \frac{.35m(m-k)}{\binom{m}{2}} \geq \frac{.7(m-k)}{m} \geq \frac{.7(m-k)(m-(k+1))}{m^2}. \tag{11}$$

Explanation: As in the previous case, there are $m-k$ choices for an edge $e = (v, w) \in M \setminus E(B)$. Then there are at least $2(\delta_B - 1) - (m-1)$ pairs of vertices (a, b) such that $(a, b) \in M$ and $(v, b) \in E(B)$, $(a, w) \in E(B)$ and at least $2(\delta_B - 1) - (m-1) - k \geq .72m - 1$ pairs $(a, b) \in M \setminus E(B)$, in which case $M' = (M \cup (a, w), (v, b)) \setminus ((v, w), (a, b))$ is a member of $\mathcal{M}_{k+2}$ and $(M, M') \in \mathcal{X}$. Now, every such pair of edges$\{(v, w), (a, b)\}$ gets counted twice. Hence, the actual number of pairs is at least $.35m$ and (11) follows.

It follows that $\mu_m \geq (.46)^k m^{-k} k! \mu_{m-k}$ for $k \geq 0$ and so

$$m! = \mu_0 + \mu_1 + \cdots + \mu_m \leq \mu_m \sum_{k=0}^{m} \frac{m^k}{(.46)^k k!} \leq \mu_m e^{m/.46}.$$

□

6.2 Upper bound for $|T_\Gamma|/|S_\Gamma|$

We will now extend an approach of Dyer, Frieze, and Jerrum [10] to directed graphs.

Let $k^* = \lfloor 7 \ln m \rfloor$, and for $1 \leq k \leq m/2$, define $g(k) = m^7 k! (7 \ln m)^{-k}$ and

$$f(k) = \begin{cases} g(k), & \text{if } k \leq k^*, \\ g(k^*), & \text{otherwise.} \end{cases}$$

Lemma 6. *Let f be the function defined above. Then*

(a) f is nonincreasing and satisfies

$$f(k-1) \geq 7k^{-1} f(k) \ln m;$$

(b) $f(k) \geq 1$, for all k.

Proof Observe that g is unimodal and that k^* is the value of k minimizing $g(k)$; it follows that f is non-increasing. When $k \leq k^*$, we have $f(k-1) = g(k-1) = (7 \ln m)k^{-1} g(k) = (7 \ln m)k^{-1} f(k)$. Otherwise, $f(k-1) = g(k^*) = f(k) \geq (7 \ln m)k^{-1} f(k)$. In either case, the inequality in part (a) of the lemma holds.

Part (b) of the lemma follows from the chain of inequalities.

$$\frac{1}{f(k)} \leq \frac{1}{g(k^*)} = \frac{(7 \ln m)^{k^*}}{m^7 k^*!} \leq m^{-7} \sum_{k=0}^{\infty} \frac{(7 \ln m)^k}{k!} = m^{-7} \exp(7 \ln m) = 1.$$

Lemma 7.

$$|S_\Gamma| \geq m^{-8} |T_\Gamma|.$$

Proof For $1 \leq k \leq \lfloor m/2 \rfloor$, let Φ_k be the set of all PD's in Γ containing exactly k cycles, and let $\Phi = \cup_k \Phi_k$ be the set of all PD's. Define

$$\Psi = \{(F, F') : F \in \Phi_k, F' \in \Phi_{k-1} \text{and} F \oplus F' \cong C_0\},$$

where $\oplus$ denotes symmetric difference and C_0 is the directed graph on four vertices with two vertices having in-degree two, out-degree zero and two vertices having out-degree two, in-degree zero. Observe that if $(F, F') \in \Psi$ then F' can be obtained from F by deleting two edges and adding two edges and that this operation reduces the number of cycles by exactly one.

Our proof strategy is to define a positive weight function on the arc set Ψ such that the total weight of arcs leaving each node (PD) $F \in \Phi \setminus \Phi_1$ is at least one greater than the total weight of arcs entering F. This will imply that the total weight of arcs entering Φ_1 is an upper bound on the number of non-Hamiltonian PD's in Γ, and that the maximum total weight of arcs entering a single node in Φ_1 ia an upper bound on the ratio $|\Phi \setminus \Phi_1|/|\Phi_1|$.

The weight function $w : \Psi \to R^+$is defined as follows: For any arc (F, F') with $F' \in \Phi_k$, if the PD F' is obtained from F by coalescing two cycles of length γ_1 and γ_2 into a single cycle of length $\gamma_1 + \gamma_2$, then $w(F, F) = (\gamma_1^{-1} + \gamma_2^{-1}) f(k)$.

Let $F \in \Phi_k$ be a PD with $k > 1$ cycles $C_1, C_2, ..., C_k$, of lengths $\gamma_1, \gamma_2, ..., \gamma_k$. Let us try to obtain a lower bound on the weight of the arcs going out of F. Suppose we chose cycle C_i to be one of the two cycles that coalesces with the other to form a new cycle. The number of ways to pick an edge from C_i, is γ_i. Having picked an edge (u, v) from C_i, we need to find an edge (s, t) on the remaining cycles such that (u, t) and (s, v) are edges in the digraph Γ. Since the out-degree of u is at least $.99m$, we have at least $.99m - (\gamma_i - 1)$ possible choices for s. Similarly, we have at least $.99m - (\gamma_i - 1)$ possible choices for t. The number of feasible edges (s, t) is

$$\begin{aligned} &\geq .99m - (\gamma_i - 1) + .99m - (\gamma_i - 1) - (m - \gamma_i) \\ &> .98m - \gamma_i \end{aligned}$$

for each edge in C_i. The total number of ways to form a new cycle using C_i is thus at least $\gamma_i(.98m - \gamma_i)$. The weight of any arc leaving F is at least $\gamma_i^{-1} f(k-1)$, which, by Lemma 2, is bounded below by $(7 \ln m)(k\gamma_i)^{-1} f(k)$. We also have to note that every pair $(u, v), (s, t)$ may get counted twice, once from each cycle. Thus, the total weight of arcs leaving F is bounded as follows:

$$\begin{aligned}\sum_{F^+:(F,F^+)\in\Psi} w(F,F^+) &\geq \frac{1}{2}\sum_{i=1}^{k}\gamma_i[.98m-\gamma_i]\frac{(7\ln m)f(k)}{k\gamma_i}\\ &= \frac{1}{2}m[.98k-1]\frac{(7\ln m)f(k)}{k}\\ &\geq 3f(k)m\ln m \qquad (12)\end{aligned}$$

where we have used the fact that $k \geq 2$.

We now give an upper bound to the weight of the arcs $(F^-, F) \in \Psi$ entering F. Suppose once again that F has cycles $C_1, C_2, ..., C_k$, of lengths $\gamma_1, \gamma_2, ..., \gamma_k$. A directed arc from F^- to F implies that one of the cycles of F was formed by coalescing two cycles of F^-. Suppose C_i was formed by coalescing two cycles from F^-. By removing the added edges from C_i and putting back the deleted edges, we can generate F^-. Let a and b be the lengths of the corresponding cycles in F^-. Remove one of the added edges from C_i and go around the cycle and remove the edge which is at a distance a. This way we make sure that the deletion of the first of the two added edges uniquely determines the two paths of lengths $a-1$ and $b-1$. Since the added edge in F could be any of the γ_i edges, we have at most γ_i choices.

The total weight of arcs entering F can thus be bounded above as follows:

$$\begin{aligned}\sum_{F^-:(F^-,F)\in\Psi} w(F^-,F) &\leq \sum_{i=1}^{k}\gamma_i f(k)\sum_{\substack{a,b\geq 1\\ a+b=\gamma_i}}\left(\frac{1}{a}+\frac{1}{b}\right)\\ &= \sum_{i=1}^{k}\gamma_i f(k)\sum_{a=1}^{\gamma_i-1}\left(\frac{1}{a}+\frac{1}{\gamma_i-a}\right)\\ &\leq 2f(k)mH(m) \qquad (13)\end{aligned}$$

where $H_m = \sum_{i=1}^{m} i^{-1} \leq \ln m + 1$ is the mth harmonic number. Combining inequalities (12) and (13), we have

$$\begin{aligned}\sum_{F^+:(F,F^+)\in\Psi} w(F,F^+) - \sum_{F^-:(F^-,F)\in\Psi} w(F^-,F) &\geq 3f(k)m\ln m - 2f(k)mH(m)\\ &\geq f(k)m(\ln m-2)\\ &\geq m(\ln m-2)\end{aligned}$$

where the final inequality is by Lemma 6. Thus the total weight of arcs leaving F exceeds the total weight of arcs entering by at least 1, provided $m \geq e^3$. The number of non-Hamiltonian PD's $|\Phi \setminus \Phi_1|$ is bounded above by the total weight of arcs entering Φ_1, which in turn is bounded - see inequality (13) - by $|\Phi_1| \times 2f(1)mH_m \leq m^8|\Phi_1|/2$. $\square$

6.3 Lower Bound on $|R_\Gamma|$:

In this section, we will use an argument similar in flavor to the argument of the previous section to give a lower bound on $|R_\Gamma|$. Each $\rho \in S_\Gamma$ yields another permutation $\lambda = \lambda(\rho) = \phi\rho$. Recall that ϕ is considered to be fixed and $\rho \in R_\Gamma$ iff $\lambda = \phi\rho$ is cyclic.

We show next that if $\rho \in S_\Gamma$ then λ has an odd number of cycles. Now ϕ is even and so λ and ρ have the same sign, $(-1)^{m-1}$. If λ has cycles $C_1, C_2, \ldots, C_k$ of sizes $\gamma_1, \gamma_2, \ldots, \gamma_k$ then the sign of λ is $(-1)^{\gamma_1 - 1 + \cdots + \gamma_k - 1} = (-1)^{m-k} = (-1)^{m-1}$.

We let

$$k_0 = 2\lfloor 50001 \ln m \rfloor + 1$$

and we will consider the following conditions for ρ: (i) λ must have at least k_0 cycles, (ii) the longest cycle in λ has length at most $\leq .8m$, and (iii) the sum of the lengths of the longest two cycles is at most $\leq .92m$.

The λ with less than k_0 cycles will be considered later in the section. Given (i), it is not clear whether or not we can find ρ satisfying conditions (ii) and (iii). Let S_k, k odd, be the set of all $\rho \in S_\Gamma$ such that $\lambda(\rho)$ has $k \geq k_0$ cycles. We define a partition $S_k = P_k \cup Q_k \cup R_k$. Let P_k be the set of all ρ which do not satisfy condition (ii), Q_k be the set of those which satisfy (ii) but not (iii) and R_k be the set of those which satisfy both (ii) and (iii).

We will first show that a constant fraction satisfying condition (i) also satisfy conditions (ii) and (iii).

6.3.1 $|P_k|/|Q_k \cup R_k| \leq 1/60$

We define $X_k = \{(\rho_1, \rho_2) : \rho_1 \in P_k, \rho_2 \in Q_k \cup R_k$ and $\Lambda_1 \oplus \Lambda_2 \cong H_0\}$ where $\Lambda_i = PD_{\lambda(\rho_i)}, i = 1, 2$ and $\oplus$ stands for symmetric difference and H_0 is the directed graph on six vertices as in Figure 1(c). The weight function $w : X_k \to R^+$ is defined as follows: If Λ_2 is obtained by breaking and patching two cycles of Λ_1 of length γ_1 and γ_2 into two cycles of length γ' and γ'', the weight $w(\rho_1, \rho_2) = \gamma_1^{-1} + \gamma_2^{-1}$. We restrict X_k to those (ρ_1, ρ_2) for which $\gamma', \gamma'' \geq m/4$.

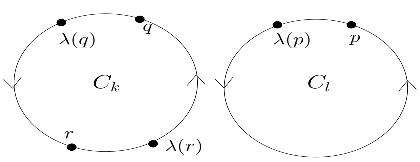

Figure 1(a)

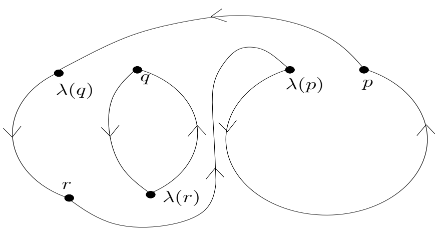

Figure 1(b)

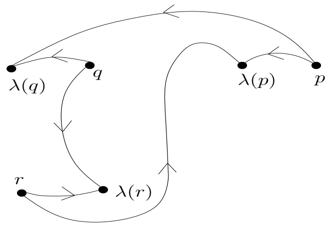

Figure 1(c)

Suppose $\rho_1 \in P_k$ and $C_1, C_2, ..., C_k$ are the cycles of Λ_1 in increasing order of size. Let γ_t denote the length of cycle C_t for $t \in [k]$. We will combine C_k with a smaller cycle C_ℓ , $\ell \neq k$, to obtain two cycles (as in Figure 1(b)) such that the new auxiliary graph constructed, say $\Lambda_2 = \Lambda(\rho_2)$, belongs to $Q_k \cup R_k$. The weight of (ρ_1, ρ_2) is $\gamma_k^{-1} + \gamma_\ell^{-1}$. While combining the two cycles, we will have to ensure that ρ_2 is a cyclic permutation.

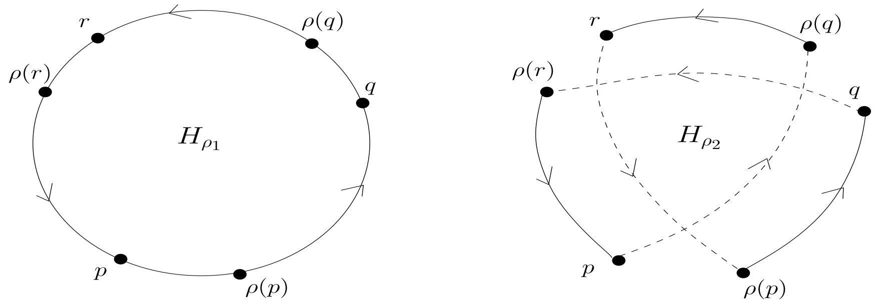

Figure 2

To create $\rho_2 \in Q_k \cup R_k$ we delete edges $(p, \lambda(p)), (q, \lambda(q))$ and $(r, \lambda(r))$, from Λ_1 where p, q, r appear in this order on the Hamilton cycle H_{ρ_1} of Γ, corresponding to ρ_1, and replace them with the edges $(p, \lambda(q)), (q, \lambda(r))$ and $(r, \lambda(p))$ as shown in Figure 2 to maintain Hamiltonicity. Vertices q and r are chosen on cycle C_k and vertex p is chosen on C_ℓ, $\ell \neq k$. We will of course have to impose the restriction on p, q and r in this construction that $(p, \rho(q)), (q, \rho(r))$ and $(r, \rho(p))$ are all edges of Γ. There will be other restrictions necessary.

The number of choices for p is γ_ℓ. Given p we rule out at most $m - \gamma_k$ choices for q due to not being on C_k. Recall that C_k is the longest cycle of Λ_1 and it corresponds to $\rho_1 \in P_k$ and so $\gamma_k \geq .8m$. This leaves us with at least $.99m + .8m - m = .79m$ choices. Having chosen q we rule out $m/2$ choices for r within distance $\leq m/4$ from q on C_k. For each such choice, both of the two new cycles replacing C_k, C_ℓ are of size $\geq m/4$, placing ρ_2 in $Q_k \cup R_k$.

Some of these choices are inadmissable because we need to ensure that ρ_2 is a cyclic permutation. First we will restrict our choice of q to one of the first $m/4$ vertices following p on H_{ρ_1}. We have at least $m/4 - .21m \geq .04m$ such choices. Now we must choose r from the remaining $3m/4$ vertices. We have ruled out $m/2$ already and we rule out a further $\leq .02m$ choices of r for which r is not an in-neighbour of $\lambda(p)$ or $\lambda(r)$ is not an out-neighbour of q. This yields at least $(.75 + .8 - .5 - 1 - .02)m = .03m$ choices.

Thus

$$\begin{aligned}
\sum_{\rho_2:(\rho_1,\rho_2)\in X_k} w(\rho_1,\rho_2) &\geq \sum_{\ell=1}^{k-1} \gamma_\ell[(.04m)(.03m)](\gamma_k^{-1}+\gamma_\ell^{-1}) \\
&\geq .0012\sum_{\ell=1}^{k-1}\gamma_\ell m^2(m^{-1}+\gamma_\ell^{-1}) \\
&= .0012\sum_{\ell=1}^{k-1}[\gamma_\ell+m]m \\
&= .0012[m-\gamma_k+(k-1)m]m \\
&\geq .0012(k-1)m^2 \\
&\geq 120m^2\ln m \qquad (14)
\end{aligned}$$

We now obtain an upper bound on the total weight of pairs containing ρ_2 of $Q_k \cup R_k$. Suppose that $C_1', C_2', \ldots, C_k'$ are the cycles of Λ_2 in increasing order of length $\gamma_1' \leq \gamma_2' \leq \cdots \leq \gamma_k'$. Our choices for Λ_1 are restricted as follows: We must choose two cycles C_r', C_s' such that $\gamma_r' + \gamma_s' = \gamma_k + \gamma_\ell \geq \gamma_k \geq .8m$ and $\gamma_r', \gamma_s' \geq m/4$. This implies $\{r,s\} = \{k-1,k\}$. Given this, we see that the total weight of edges entering Λ_2 can be bounded as follows: The parameter a is the length of the path from q to r on C_k.

$$\begin{aligned}
\sum_{\rho_1:(\rho_1,\rho_2)\in X_k} w(\rho_1,\rho_2) &\leq m^2 \max_{M\leq m}\left\{\sum_{\substack{a+b=M\\ a,b\geq 2}}\left(\frac{1}{a}+\frac{1}{b}\right)\right\} \\
&\leq m^2\sum_{a=2}^{m}\frac{2}{a} \\
&\leq 2m^2\ln m \qquad (15)
\end{aligned}$$

It follows that

$$120m^2|P_k|\ln m \leq \sum_{(\rho_1,\rho_2)\in X_k} w(\rho_1,\rho_2) \leq 2m^2|Q_k\cup R_k|\ln m \qquad (16)$$

We have thus shown that

$$|P_k| \leq |Q_k \cup R_k|/60. \qquad (17)$$

6.3.2 $|Q_k|/|R_k| \leq 1/200$

The argument that follows is similar to the one used above to show that $|Q_k \cup R_k|$ is a constant fraction of $|S_k|$. Now let $Y_k = \{(\rho_1,\rho_2) : \rho_1 \in Q_k, \rho_2 \in R_k \text{ and } \Lambda_1 \oplus \Lambda_2 \cong H_0\}$ where H_0 is the same directed graph on six vertices as in Figure 1(c). The weight function $w : Y_k \to R^+$ is defined as follows: If Λ_2 is obtained by breaking and patching two cycles of Λ_1 of length γ_i and γ_j into two cycles of length γ' and γ'', then $w(\rho_1,\rho_2) = \gamma_i^{-1} + \gamma_j^{-1}$. We again restrict our attention to (ρ_1,ρ_2) such that $\gamma', \gamma'' \geq m/10$.

Suppose $\rho_1 \in Q_k$ and $C_1, C_2, ..., C_k$ are the cycles of Λ_1 in the order of increasing size $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_k$ where $\gamma_k \leq .8m$ and $\gamma_k + \gamma_{k-1} > .92m$. We will combine C_k with a smaller cycle C_ℓ , $\ell \notin \{k, k-1\}$, to obtain two cycles (as in Figure 1(b)) such that the new auxiliary graph constructed, Λ_2, belongs to R_k.

While q and r are chosen from cycle C_k, p is chosen from C_ℓ. The number of choices for p is γ_ℓ. Since $\gamma_k + \gamma_{k-1} \geq .92m$, we have $\gamma_k \geq .46m$. The number of *feasible* choices for q and r is therefore at least $.99m + .46m - m = .45m$. Having chosen q we rule out $.2m$ choices for r within distance $\leq .1m$ from q on C_k. For each such choice, the size of the new largest cycle is at most $\max\{\gamma_{k-1}, \gamma_k + .08m - .1m\}$ and both of the two new cycles replacing C_k, C_ℓ are of size $\geq .1m$, placing Γ_2 in R_k.

Some of these choices are inadmissable because we need to ensure that ρ_2 is a cyclic permutation. First we will restrict our choice of q so that q is one of the first $.1m$ feasible C_k-vertices following p on H_ρ. Now we must choose r from the remaining $\geq .35m$ feasible C_k-vertices. We have ruled out $.2m$ already and we rule out a further $\leq .02m$ choices of r for which r is not an in-neighbour of $\lambda(p)$ or $\lambda(r)$ is not an out-neighbour of q. This yields at least $.13m$ choices.

Thus,

$$\begin{aligned}
\sum_{\rho_2:(\rho_1,\rho_2)\in Y_k} w(\rho_1,\rho_2) &\geq \sum_{\ell=1}^{k-2} \gamma_\ell[(.1m)(.13m)](\gamma_k^{-1} + \gamma_\ell^{-1}) \\
&\geq .013 \sum_{\ell=1}^{k-2} \gamma_\ell m^2 (m^{-1} + \gamma_\ell^{-1}) \\
&= .013 \sum_{\ell=1}^{k-2} [\gamma_\ell + m]m \\
&= .013[m - \gamma_k - \gamma_{k-1} + (k-2)m]m \\
&\geq .013(k-2)m^2 \\
&\geq 1300 m^2 \ln m. \qquad (18)
\end{aligned}$$

We now obtain an upper bound on the total weight of pairs containing a member ρ_2 of R_k. Suppose that $C_1', C_2', \ldots, C_k'$ are the cycles of Λ_2 in increasing order of length $\gamma_1' \leq \gamma_2' \leq \cdots \leq \gamma_k'$. Our choices for Λ_1 are restricted as follows: We must choose two cycles C_r', C_s' such that $\gamma_r' + \gamma_s' = \gamma_k + \gamma_\ell \geq \gamma_k \geq .46m$ and $\gamma_r', \gamma_s' \geq .1m$. This implies $\{r, s\} \subseteq \{k-2, k-1, k\}$. (Those Λ_2 that are paired with Λ_1 have at most 3 cycles of size greater than $.08m$). Given this, we see that

$$\begin{aligned}
\sum_{\rho_1:(\rho_1,\rho_2)\in Y_k} w(\rho_1,\rho_2) &\leq 3m^2 \max_{M\leq m} \left\{ \sum_{\substack{a+b=M \\ a,b\geq 2}} \left(\frac{1}{a} + \frac{1}{b}\right) \right\} \\
&\leq 3m^2 \sum_{a=2}^{m} \frac{2}{a} \\
&\leq 6m^2 \ln m.
\end{aligned}$$

It follows that

$$1300 m^2 |Q_k| \ln m \leq \sum_{(\rho_1,\rho_2)\in Y_k} w(\rho_1,\rho_2) \leq 6m^2 |R_k| \ln m. \qquad (19)$$

From (17) and (19), we have

$$|R_k| \geq 9|S_k|/10. \qquad (20)$$

6.4 Final Estimate

We can now complete our proof of Theorem 2. Fix $k_0 \leq k \leq \lfloor m/2 \rfloor$ and let

$$Z_k = \{(\rho_1, \rho_2) : \rho_1 \in R_k, \rho_2 \in S_{k-2}, \text{ and } \rho_1 \oplus \rho_2 \cong H_1\},$$

where H_1 is the graph shown in Figure 3(c) and Figure 4(c). Here we combine two cases. We take 3 (see Figure 3) or 4 (see Figure 4) cycles from Λ_1 and reduce this number by 2 to create Λ_2.

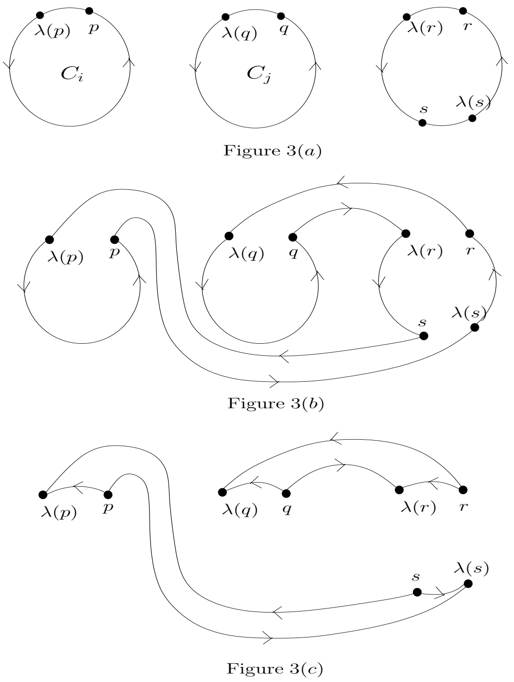

Figure (a) shows three cycles of Λ_1 and figure (b) shows the cycle of Λ_2 obtained by coalescing the three cycles of Λ_1. Figure (c) gives the graph $H_1 = \Lambda_1 \oplus \Lambda_2$.

Let $\rho_1 \in R_k$ give rise to Λ_1 that has cycles $C_1, C_2, ..., C_k$, of lengths $\gamma_1, \gamma_2, ..., \gamma_k$. For $(\rho_1, \rho_2) \in Z_k$ we will delete four edges from Λ_1 and add in four new edges to obtain Λ_2. If Λ_2 is obtained from Λ_1 by breaking and patching cycles of length $\gamma_1, \gamma_2, ..., \gamma_s$, then $w(\Lambda_1, \Lambda_2) = (\gamma_1^{-1} + \gamma_2^{-1} + ... + \gamma_s^{-1})$. We will first obtain a lower bound on the total weight of pairs containing ρ_1.

We choose cycles C_i and C_j from Λ_1 and delete one edge from each, $(p, \lambda(p))$ and $(q, \lambda(q))$, respectively. The remaining two edges to be deleted should lie outside the cycles, C_i and C_j. Let the edges deleted from outside the cycles, C_i and C_j, be $(r, \lambda(r))$ and $(s, \lambda(s))$.

We will give a lower bound on the number of sets of four edges that can be deleted from the auxiliary graph. Suppose $(p, \lambda(p))$ and $(q, \lambda(q))$ are chosen from C_i and C_j, respectively. The number of choices for $(p, \lambda(p))$ and $(q, \lambda(q))$ are γ_i and γ_j, respectively. Now, we remove two edges, $(r, \lambda(r))$ and $(s, \lambda(s))$, outside of cycles γ_i and γ_j such that edges of the form $(x, \lambda(y))$ and edges of the form $(y, \lambda(x))$ are edges in Γ for $x \in \{p, q\}$ and $y \in \{r, s\}$. The number of choices for $(r, \lambda(r))$ such that $(p, \lambda(r))$ is an edge in Γ is at least $.99m$. The number of choices for $(r, \lambda(r))$ such that both $(p, \lambda(r))$ and $(q, \lambda(r))$ are edges of in Γ is at least $.99m + .99m - m \geq .98m$. By a similar argument, we have that the number of choices for $(r, \lambda(r))$ such that both $(r, \lambda(p))$ and $(r, \lambda(q))$ are edges of Γ is at least $.98m$. Thus the number of choices for $(r, \lambda(r))$ is at least

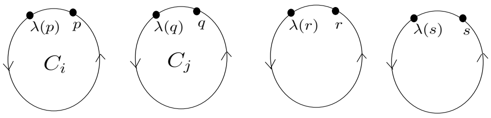

Figure 4(a)

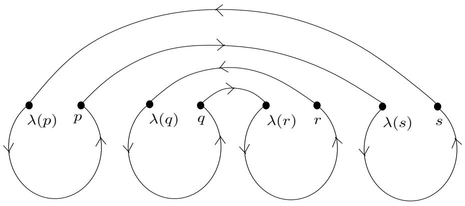

Figure 4(b)

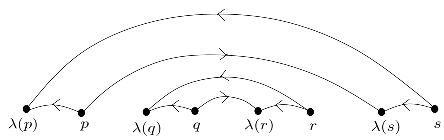

Figure 4(c)

Figure (a) shows four cycles of Λ_1 and figure (b) shows the two cycles of Λ_2 obtained by coalescing the four cycles of Λ_1. Figure (c) gives the graph $H_1 = \Lambda_1 \oplus \Lambda_2$

$.98m + .98m - m \geq .96m$. Since $(r, \lambda(r))$ should lie outside of cycles C_i and C_j, the number of choices for $(r, \lambda(r))$ is $.96m - \gamma_i - \gamma_j$. The same applies to $(s, \lambda(s))$. We also add the extra condition that p, q, r, s lie in the order p, q, r, s or p, s, r, q on H_{ρ_1}, (see Figure 5) thereby ensuring that ρ_2 is a cyclic permutation. The number of choices for r and s is at least $\binom{(.96m-\gamma_i-\gamma_j)/2}{2}$. The number of choices for p, q, r, s is therefore at least

$$\gamma_i\gamma_j\binom{(.96m-\gamma_i-\gamma_j)/2}{2} \geq .0001\gamma_i\gamma_j m^2 \tag{21}$$

We can pick the same set of four edges at most 12 times by picking one of the four edges to be the first edge and one of the remaining three to be the second edge from a different cycle. The weight of (Λ_1, Λ_2) is at least $\gamma_i^{-1} + \gamma_j^{-1}$. Consequently, the weight of arcs emanating from $\rho_1 \in R_k$ is

$$\begin{aligned}
&\geq \frac{m^2}{12000}\sum_{i=1}^{k}\sum_{j\neq i}\gamma_i\gamma_j\left(\frac{1}{\gamma_i}+\frac{1}{\gamma_j}\right) \\
&= \frac{m^2}{12000}\sum_{i=1}^{k}\sum_{j\neq i}(\gamma_i+\gamma_j) \\
&\geq \frac{(k-1)m^3}{12000} \\
&> 8m^3\ln m
\end{aligned} \tag{22}$$

Now, we will give an upper bound on the total weight of pairs $(\Lambda_1, \Lambda_2) \in Z_k$ for a fixed $\Lambda_2 \in S_k$.

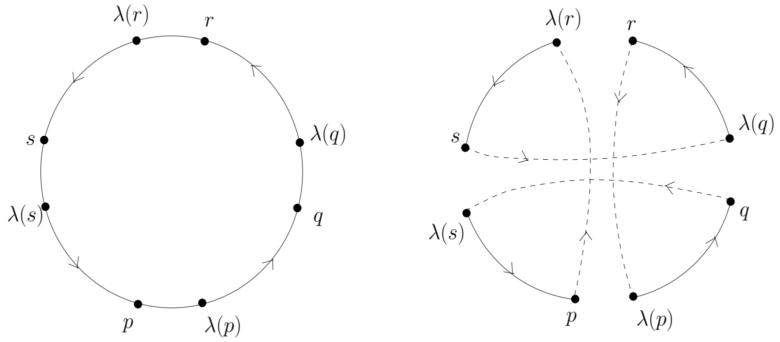

Figure 5

Suppose Λ_2 has cycles $C_1, C_2, ..., C_k$, of lengths $\gamma_1, \gamma_2, ..., \gamma_k$. Either one cycle of Λ_2 was formed by coalescing three cycles of Λ_1 or two cycles of Λ_2 were formed by coalescing four cycles of Λ_1 into two.

Case 1: Suppose $C_i \in \Lambda_2$ was formed by coalescing three cycles of lengths a, b and c from Λ_1 with one edge deleted from the cycles of length a and b and two edges deleted from the cycle of length c, as in Figure 3. There is a choice $1 \leq d \leq c-1$ for length of the *first* of the paths created from the cycle of length c. Given a, b, c, d, there are γ_i choices for p say. Thus the total weight of pairs containing a fixed Λ_2 can be bounded from above in this case by

$$\begin{aligned}
&\sum_{i=1}^{k} \gamma_i \sum_{\substack{(a,b,c,d):a,b,c\geq 1\\ 1\leq d\leq c-1\\ a+b+c=\gamma_i}} \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\\
&\leq m \sum_{i=1}^{k} \gamma_i \sum_{\substack{(a,b,c):a,b,c\geq 1\\ a+b+c=\gamma_i}} \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\\
&\leq 3m \sum_{i=1}^{k} \gamma_i^2 (\ln \gamma_i + 1)\\
&\leq 4m^3 \ln m \qquad (23)
\end{aligned}$$

Case 2: Suppose $C_i, C_j \in \Lambda_2$ were formed by combining four cycles of lengths a, b, c and d from Λ_1 with one edge deleted from each of the four cycles, as in Figure 4. The total weight of pairs containing a fixed Λ_2 can be bounded from above in this case by

$$\begin{aligned}
&\sum_{i=1}^{k}\sum_{j=i+1}^{k}\gamma_i\gamma_j \sum_{\substack{(a,c):a,c\geq 2\\ a+c=\gamma_i}} \sum_{\substack{(b,d):b,d\geq 2\\ b+d=\gamma_j}} \left(\frac{1}{a}+\frac{1}{c}+\frac{1}{b}+\frac{1}{d}\right)\\
&=\frac{1}{2}\sum_{i=1}^{k}\sum_{j\neq i}\gamma_i\gamma_j\sum_{a=2}^{\gamma_i-2}\sum_{b=2}^{\gamma_j-2}\left(\frac{1}{a}+\frac{1}{\gamma_i-a}+\frac{1}{b}+\frac{1}{\gamma_j-b}\right)\\
&\leq \frac{1}{2}\sum_{i=1}^{k}\sum_{j\neq i}\gamma_i\gamma_j(2\gamma_j\ln\gamma_i+2\gamma_i\ln\gamma_j)\\
&\leq \frac{1}{2}\sum_{i=1}^{k}4\gamma_i m^2\ln m\\
&=2m^3\ln m.
\end{aligned} \tag{24}$$

Combining inequalities (22), (23) and (24), we have that for $k \geq k_0$,

$$(6m^3\ln m)|S_{k-2}| \geq w(Z_k) \geq (8m^3\ln m)|R_k|$$

and hence

$$|S_{k-2}| \geq 4|R_k|/3 \geq 6|S_k|/5$$

after using (20). It follows that

$$\sum_{\substack{k\geq k_0\\ k \text{ odd}}} |S_k| \leq 6|S_{k_0-2}|. \tag{25}$$

Recall that all λ have an odd number of cycles.

We will now consider λ with at most $3 \leq k \leq k_0 - 2$ cycles. We will show that

$$\frac{|S_{k-2}|}{|S_k|} \geq \frac{1}{m^4} \tag{26}$$

which implies

$$|S_k| \leq m^{2(k-1)}|S_1|.$$

Hence,

$$\frac{|R_\Gamma|}{|S_\Gamma|} = \frac{|S_1|}{\sum_{\substack{k\geq 1\\ k \text{ odd}}} |S_k|} = \Omega(m^{-2k_0}).$$

Therefore we have

$$|R_\Gamma| = e^{-o(m)}|S_\Gamma| \geq e^{-o(m)}|T_\Gamma| \geq m!e^{-(5/2+o(1))m}.$$

It remains to prove (26).

Fix $k \leq k_0 - 2$ and let P_k, Q_k and R_k partition S_k as before. Suppose $\Lambda \in S_k$ has cycles $C_1, C_2, \ldots, C_k$ in increasing order of size $\gamma_1, \gamma_2, \ldots, \gamma_k$.

Suppose $\rho \in S_k$. Suppose $(p, \lambda(p)) \in C_i$ and $(q, \lambda(q)) \in C_j$ are deleted where $i \neq j$ and $i, j \in \{1, 2, ..., k-1\}$. We delete two edges $(r, \lambda(r))$ and $(s, \lambda(s))$ from C_k such that the four paths created can be patched into one cycle, reducing the number of cycles by two. We also ensure that the corresponding ρ' is cyclic. If $\rho \in R_k$ then $\gamma_i + \gamma_j \leq .92m$. If $\rho \in P_k \cup Q_k$ then $\gamma_i + \gamma_j \leq m - \gamma_k \leq .54m$. The binomial term in (21) is thus at least $\binom{(.96m-.92m)/2}{2} > 0$ and this ensures that we have at least one pair of edges in γ_k that can be deleted. Thus there is at least one way to transform $\rho \in S_k$ into $\rho' \in S_{k-2}$. Fix one way for each $\rho \in S_k$. Clearly, each ρ' arises in at most m^4 times in this way.

This verifies (26) and completes our proof of Thereom 2. □

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