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Deterministic and probabilisitic analysis of algorithms
Global optimization

Let me stress, however, that my co-editors are not 'area' editors - each will handle papers in all topics covered by the editorial policy.

The editorial structure will remain as before. Briefly, the editor-in-chief retains overall control of *Mathematical Programming* and *Mathematical Programming Studies* while the co-editors make editorial decisions on individual papers. Manuscripts sent to the editor-in-chief will frequently be reassigned to a co-editor. As always, we remain committed to as prompt a refereeing process as possible. In the coming months we may be making minor stylistic changes - please consult a current copy of the 'Instructions to Authors' before submitting your papers.

Finally, we note the sad loss of another of our senior editors, Dr. E. M. L. Beale. An obituary notice will appear in the Mathematical Programming Society's newsletter, *Optima*.

Michael J. Todd

Mathematical Programming 35 (1986) 3-16
North-Holland

ON LINEAR PROGRAMS WITH RANDOM COSTS

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Received 9 April 1985

We consider linear programs in which the objective function (cost) coefficients are independent non-negative random variables, and give upper bounds for the random minimum cost. One application shows that for quadratic assignment problems with such costs certain branch-and-bound algorithms usually take more than exponential time.

Key words: Random linear program, probabilistic analysis of algorithms, average complexity, quadratic assignment problem, branch-and-bound.

1. Introduction

This paper is concerned with a probabilistic analysis of linear programs with random objective function coefficients. In particular we consider the problem

$$\begin{aligned} &\text{minimise} \quad \sum_{j=1}^{n} c_j x_j \\ &\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1, 2, \ldots, m, \qquad (1.1) \\ &\qquad\qquad x_j \geq 0, \quad j = 1, 2, \ldots, n. \end{aligned}$$

We assume that $c_1, c_2, \ldots, c_n$ are independent non-negative random variables. The remaining parameters of (1.1), that is the a_{ij}'s and b_i's, are assumed to be known constants.

In order to give a flavour of our results we quote the following theorem which will be proved later. Let z^* denote the (random) minimum value of (1.1).

Theorem 1.1. *Suppose that $c_1, c_2, \ldots, c_n$ are independent uniform* [0, 1] *random variables and $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n$ is any* fixed *feasible solution to* (1.1) (*not necessarily optimal*). *Then*

$$E(z^*) \leq m \max\{\hat{x}_j : j = 1, 2, \ldots, n\}. \qquad (1.2)$$

This result will be shown to generalise a recent result of Karp [7] which states that the expected value of a random assignment problem with independent uniform [0, 1] costs is no more than 2. An earlier bound of 3 had been obtained by Walkup [9]. In Section 2 below we prove a somewhat stronger result than Theorem 1.1 together with results on the probability that z^* exceeds the right hand side of (1.2) by a significant amount.

Interesting though such results on the expected optimal value may be in themselves, they also have an important impact in algorithmic analysis. Branch-and-bound is an important technique for solving NP-hard discrete optimisation problems exactly. In many cases the bounds used are based on LP relaxations of the original problem. It is important to know how close this bound is to the value of the original problem, on average.

Results like Theorem 1.1 are obviously useful in such circumstances. In Section 3 below we consider the probable effectiveness of branch-and-bound algorithms for solving one particular important problem—the Quadratic Assignment Problem. Our main result is that for a certain natural stochastic model, with probability tending to one, any branch-and-bound algorithm based on proposed LP bounds takes super-exponential time.

2. Main results

We first prove a result that contains Theorem 1.1 as a special case.

Theorem 2.1. *Let $c_1, c_2, \ldots, c_n$ be independent non-negative random variables. Suppose that there exists β, $0<\beta\leqslant 1$ such that, for $j=1,2,\ldots,n$,*

$$E(c_j \mid c_j \geqslant h) \geqslant E(c_j)+\beta h \tag{2.1}$$

for all $h>0$ with $P(c_j\geqslant h)>0$. Let $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n$ be any fixed feasible solution to (1.1). Then, assuming $E(c_1)\hat{x}_1 \geqslant E(c_2)\hat{x}_2 \geqslant \cdots \geqslant E(c_n)\hat{x}_n$,

$$E(z^*) \leqslant \beta^{-1} \sum_{i=1}^{m} E(c_i)\hat{x}_i. \tag{2.2}$$

Proof. Let $c\in R^n$ be the vector of c_j's. Let $a_j=(a_{1j},\ldots,a_{mj})^{\mathrm{T}}$ be the jth column of the constraint matrix of (1.1) for $j=1,2,\ldots,n$. Let $b=(b_1,b_2,\ldots,b_m)^{\mathrm{T}}$ be the vector of right hand sides.

We can assume without loss of generality that the constraint matrix is of full row rank m and that (1.1) is non-degenerate. Otherwise we can delete constraints and/or apply standard perturbation techniques to ensure this.

Let P be the (simple) polyhedral feasible region to (1.1). Suppose P has N vertices. Let $S_1, S_2, \ldots, S_N$ be the basic index sets for the N feasible bases corresponding to the vertices. Let B_r be the basis matrix with columns a_j $(j\in S_r)$ and

$\gamma^{(r)}$ be the row vector with elements c_j $(j\in S_r)$. Then the condition that B_r is an optimal basis to (1.1) with value $z^*(c)=\gamma^{(r)}B_r^{-1}b$ is

$$c_j-\gamma^{(r)}B_r^{-1}a_j\geqslant 0 \quad (j\notin S_r). \tag{2.3}$$

This is a set of linear inequalities in the c_j's and it defines a convex polyhedral subset Q_r of R_+^n. The Q_r $(r=1,2,\ldots,N)$ partition R_+^n (except for overlapping on null-sets). Now if $c\in Q_r$ (i.e. B_r is optimal), there is only one inequality involving each $j\notin S_r$. Thus conditional on $c\in Q_r$, and on $\gamma^{(r)}$, the c_j $(j\notin S_r)$ are independent and conditioned only by (2.3). Let E_r be the event $\{c\in Q_r\}$. Then, for $j\notin S_r$,

$$E(c_j\mid E_r,\gamma^{(r)})=E(c_j\mid c_j\geqslant \gamma^{(r)}B_r^{-1}a_j,\gamma^{(r)})\geqslant \bar{c}_j+\beta\gamma^{(r)}B_r^{-1}a_j$$

where $\bar{c}_j=E(c_j)$ for $j=1,2,\ldots,n$.

Thus, for $r=1,2,\ldots,N$,

$$\begin{aligned}E\left(\sum_{j=1}^{n} c_j\hat{x}_j \,\middle|\, E_r,\gamma^{(r)}\right)&=\sum_{j\in S_r}\gamma_j^{(r)}\hat{x}_j+\sum_{j\notin S_r}E(c_j\mid E_r,\gamma^{(r)})\hat{x}_j\\&\geqslant \sum_{j\in S_r}\gamma_j^{(r)}\hat{x}_j+\sum_{j\notin S_r}(\bar{c}_j+\beta\gamma^{(r)}B_r^{-1}a_j)\hat{x}_j\\&=\sum_{j\in S_r}\gamma_j^{(r)}\hat{x}_j+j\sum_{j\notin S_r}\bar{c}_j\hat{x}_j+\beta\gamma^{(r)}B_r^{-1}\left(b-\sum_{j\in S_r}a_j\hat{x}_j\right)\end{aligned}$$

(using the fact that $\sum_{j=1}^{n} a_j\hat{x}_j=b$)

$$=\sum_{j\in S_r}(\gamma_j^{(r)}-\beta\gamma^{(r)}B_r^{-1}a_j)\hat{x}_j+\sum_{j\notin S_r}\bar{c}_j\hat{x}_j+\beta E(z^*\mid E_r,\gamma^{(r)})$$

Now, for $j\in S_r$, $\gamma_j^{(r)}=\gamma^{(r)}B_r^{-1}a_j$ and since $\gamma_j^{(r)}$, $1-\beta$ and $\hat{x}_j$ are all non-negative, we have

$$E\left(\sum_{j=1}^{n} c_j\hat{x}_j \,\middle|\, E_r,\gamma^{(r)}\right)\geqslant \sum_{j\notin S_r}\bar{c}_j\hat{x}_j+\beta E(z^*\mid E_r,\gamma^{(r)})$$

and hence

$$E\left(\sum_{j=1}^{n} c_j\hat{x}_j \,\middle|\, E_r\right)\geqslant \sum_{j\notin S_r}\bar{c}_j\hat{x}_j+\beta E(z^*\mid E_r).$$

Thus if $p_r=\Pr(E_r)$

$$\sum_{r=1}^{N} p_r E\left(\sum_{j=1}^{n} c_j\hat{x}_j \,\middle|\, E_r\right)\geqslant \sum_{r=1}^{N} p_r\sum_{j\notin S_r}\bar{c}_j\hat{x}_j+\beta\sum_{r=1}^{N} p_r E(z^*\mid E_r).$$

But

$$\sum_{r=1}^{N} p_r E\left(\sum_{j=1}^{n} c_j\hat{x}_j \,\middle|\, E_r\right)=E\left(\sum_{j=1}^{n} c_j\hat{x}_j\right)=\sum_{j=1}^{n}\bar{c}_j\hat{x}_j$$

and

$$\sum_{r=1}^{N} p_r E(z^*\mid E_r)=E(z^*)$$

and so

$$\beta E(z^*) \le \sum_{j=1}^{n} \bar{c}_j\hat{x}_j - \sum_{r=1}^{N} p_r \sum_{j\notin S_r} \bar{c}_j\hat{x}_j = \sum_{r=1}^{N} p_r \sum_{j\in S_r} \bar{c}_j\hat{x}_j. \tag{2.4}$$

The Theorem now follows from (2.4) and the fact that $\sum_{j\in S_r} \bar{c}_j\hat{x}_j \le \sum_{i=1}^{m} \bar{c}_i\hat{x}_i$ for $r = 1, 2, \ldots, N$ by assumption.

This proof is a simplification and generalisation of Karp's proof [7]. To prove Theorem 1.1 we note that for uniform [0, 1] random variables $\bar{c}_j = \beta = \frac{1}{2}$ in (2.1).

We will also be concerned with exponentially distributed random variables. A random variable X is exponentially distributed with parameter $\lambda > 0$ if

$$\Pr(X \le x) = 1 - e^{-\lambda x} \quad \text{for } x \ge 0.$$

This implies that

$$E(X \mid X \ge h) = \max(0, h) + \lambda^{-1} \ge h + \lambda^{-1}$$

and so (2.1) holds with $\beta = 1$ if the c_j's are exponentially distributed.

We therefore have

Corollary 2.2. *If $c_1, c_2, \ldots, c_n$ are independent exponentially distributed random variables with parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n$ is any fixed feasible solution to (1.1) then, assuming $\lambda_1^{-1}\hat{x}_1 \ge \lambda_2^{-1}\hat{x}_2 \ge \cdots \ge \lambda_n^{-1}\hat{x}_n$,*

$$E(z^*) \le \sum_{i=1}^{m} \lambda_i^{-1}\hat{x}_i.$$

Before discussing applications of these results we check that condition (2.1) is not particularly restrictive.

Lemma 2.3. *Let X be a non-negative random variable with finite mean $\mu > 0$. Then there exists $\beta > 0$ such that*

$$E(X \mid X \ge h) \ge \mu + \beta h$$

for all $h > 0$ with $\Pr(X \ge h) > 0$, if and only if

$$\liminf_{h\downarrow 0} h^{-1} \Pr(X \le h) > 0. \tag{2.5}$$

Proof. Let $F(h) = \Pr(X \le h)$.

(Only if.) Suppose that $\liminf_{h\downarrow 0} F(h)/h = 0$. Let $\beta > 0$ and let $\delta = \beta/2\mu > 0$. Then there exists $h > 0$ such that $F(h) < \delta h < \frac{1}{2}$. But then

$$E(X \mid X \ge h) \le \mu/(1 - F(h)) < \mu/(1 - \delta h) < \mu(1 + 2\delta h)$$

so that

$$E(X \mid X \ge h) < \mu + \beta h.$$

(If.) Now suppose that $\liminf_{h\downarrow 0} F(h)/h = 2\delta > 0$. Consider small h first. Let $0 < h_0 \le \mu/3$ be such that $F(h)/h \ge \delta$ for $0 < h \le h_0$. Then for such h

$$\begin{aligned} E(X \mid X \ge h) &\ge (\mu - hF(h))/(1 - F(h)) > (\mu - hF(h))(1 + F(h)) \\ &> \mu + F(h)\mu/3 \qquad \text{using } h \le \mu/3 \\ &\ge \mu + (\delta\mu/3)h. \end{aligned}$$

Thus the desired result holds for small h. But now consider $h > h_0$ with $\Pr(X \ge h) > 0$: if $h_0 < h \le 2\mu$ say, then

$$E(X \mid X \ge h) \ge \mu + (\delta\mu/3)h_0 \ge \mu + (\delta h_0/6)h$$

and if $h > 2\mu$ then

$$E(X \mid X \ge h) \ge h \ge \mu + \tfrac{1}{2}h.$$

We stress once again that the above results are still valid if we drop either the non-degeneracy assumption or the assumption that the row rank of the constraint matrix is equal to the number of constraints. They are also valid if some of the constraints are inequalities. Given a feasible solution $\hat{x}$ to $Ax \le b$, $x \ge 0$ say, we consider the restricted linear program min $c \cdot x$ subject to $Ax = A\hat{x}$, $x \ge 0$.

Examples. Suppose, for ease of exposition, that the objective function coefficients below are all independent and uniformly distributed on [0, 1].

(1) Linear program for d-dimensional matching.

$$\text{Minimise} \quad z = \sum_{i,j,\ldots=1}^{n} c_{ij\ldots} x_{ij\ldots}$$

$$\text{subject to} \quad \sum_{j,\ldots=1}^{n} x_{ij\ldots} = 1, \ \text{for } i = 1, 2, \ldots, n,$$

$$\vdots$$

$$x_{ij\ldots} \ge 0, \ \text{for } i, j, \ldots = 1, 2, \ldots, n.$$

The variables $x_{ij\ldots}$ are assumed to have d (>1) subscripts. The constraints are that summing over the variables with a fixed value for one subscript gives one.

This linear program has $m = dn$ constraints. Choose as a feasible solution $\hat{x}_{ij\ldots} = 1/n^{d-1}$ for $i, j, \ldots = 1, 2, \ldots, n$. So $\hat{x}_{(1)} = 1/n^{d-1}$ and hence, from Theorem 1.1,

$$E(z^*) \le dn/n^{d-1} = d/n^{d-2}.$$

Also, from Corollary 2.6 (below),

$$\Pr\{z^* \geq (1+o(1))d/n^{d-2}\} \to 0 \quad \text{as } n \to \infty.$$

(2) Linear program for d-dimensional assignment.

$$\text{Minimise} \quad z = \sum_{i,j,\ldots=1}^{n} c_{ij\ldots} x_{ij\ldots}$$

$$\text{subject to} \quad \sum_{i=1}^{n} x_{ij\ldots} = 1 \quad \text{for } j, k \ldots = 1, 2, \ldots, n,$$

$$\vdots$$

$$x_{ij\ldots} \geq 0 \quad \text{for } i, j, \ldots = 1, 2, \ldots, n.$$

We have the same set of variables as in example (1). The constraints are that keeping the values of all but one subscript constant and summing gives one.

This has $m = dn^{d-1}$ constraints. Choose feasible solution $\hat{x}_{ij\ldots} = 1/n$ for $i, j, \ldots = 1, 2, \ldots, n$. So $\hat{x}_{(1)} = 1/n$ and, as before from Theorem 1.1,

$$E(z^*) \leq dn^{d-1}/n = dn^{d-2},$$

and, from Corollary 2.6,

$$\Pr\{z^* \leq (1+o(1))dn^{d-2}\} \to 0 \quad \text{as } n \to \infty.$$

Either of these examples generalises Karp's bound for the assignment problem, as does the next one.

(3) Minimum weight matching in regular bipartite graphs.

Let $G = (V, E)$ be a regular bipartite graph, of vertex degree k. If edge-weights on G are chosen independently from the uniform distribution on $[0, 1]$, then the minimum weight z^* of a perfect matching on G satisfies

$$E(z^*) \leq |V|/k.$$

Proof. Since G is bipartite, the value z^* is that of the linear program

$$\text{minimise} \quad z = \sum_{e \in E} c_e x_e$$

$$\text{subject to} \quad \sum_{e \ni v} x_e = 1 \quad (v \in V),$$

$$x_e \geq 0.$$

Since G is regular of degree k, $\hat{x}_e = 1/k$ is a feasible solution to this LP. Thus, since the LP has $|V|$ constraints, the result now follows directly from Theorem 1.1. Furthermore, by Corollary 2.6,

$$\Pr(z^* \geq (1+o(1))|V|/k) \to 0 \quad \text{as } |V| \to \infty.$$

Note that this bound also applies to the *fractional* matching problem in regular nonbipartite graphs.

(4) Greedy heuristic for the 3-dimensional assignment problem. This problem is the integer programming problem with constraints as in Example (2) with $d = 3$. We shall show that the expected value of the three-dimensional assignment problem with independent uniform $[0, 1]$ weights is less than $2n \log n$. (All logarithms here are natural.)

Proof. Use a greedy 'plane-by-plane' heuristic, solving the minimum-cost matching problem in each plane on the remaining graph, when the previously selected matchings have been removed. We have $|V| = 2n$, and k successively $n, (n-1), \ldots, 1$. Thus using example (3) the expected value is at most

$$\frac{2n}{n} + \frac{2n}{n-1} + \cdots + \frac{2n}{1} < 2n(\log n + 1),$$

(using the relationship between the sum and the integral of $1/x$). This bound can be tightened slightly by noting that at the last three stages a *random* matching gives a better bound. This has expected value $\frac{1}{2}n$ which is better than $2n/3$, $2n/2$ or $2n/1$. Thus the expected value is at most

$$2n\left(\frac{1}{n} + \cdots + \frac{1}{4}\right) + n(\tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2}) < 2n(\log n + 1 - \tfrac{11}{6}) + \tfrac{3}{2}n$$

$$= 2n(\log n - \tfrac{1}{12}) < 2n \log n \quad \text{as claimed.}$$

Moreover, we have a polynomial-time heuristic which guarantees this. [Its time-complexity is $O(n^4)$, n phases each $O(n^3)$.] Corollary 2.6 again can be used to show that $\Pr(z_{IP} \geq (2+o(1))n \log n) \to 0$ as $n \to \infty$, where z_{IP} is the integer optimum value.

This problem is NP-hard (see Frieze [3]) and no heuristics are known with a *proven* good performance, in a probabilisitc sense.

We now consider the problem of bounding the probability that z^* exceeds the given upper bounds. We shall work with exponential random variables and derive a result for uniform random variables as a corollary.

The moment generating function of a random variable Z is defined to be $M_z(t) = E(\exp(tZ))$. It is a standard tool in probability theory and statistics and its use in this paper comes from the following: if Z is a non-negative random variable then

$$\Pr(Z \geq a) \leq E(e^{tZ})/e^{ta}, \quad \text{for all } t. \tag{2.10}$$

Theorem 2.4. *Suppose that $c_1, c_2, \ldots, c_m, \hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n$ are as in Corollary 2.2. Then*

$$M_{z^*}(t) \leq \prod_{j=1}^{m} \frac{\lambda_j}{\lambda_j - \hat{x}_j t}, \quad 0 \leq t < \lambda_1/\hat{x}_1.$$

Proof. We note first that an exponential random variable with parameter λ has moment generating function $\lambda/(\lambda - t)$. We write $M_j(t)$ for $\lambda_j/(\lambda_j - t)$. Letting E_r,

$\gamma^{(r)}$, S_r be as in the proof of Theorem 2.1, we have, for $0 \leq t < \lambda_1/\hat{x}_1$,

$$E\left(\exp\left(t \sum_{j=1}^{n} c_j\hat{x}_j\right) \Big| E_r, \gamma^{(r)}\right) = \exp\left(t \sum_{j\in S_r} \gamma_j^{(r)}\hat{x}_j\right) \prod_{j\notin S_r} E(\exp(tc_j\hat{x}_j) \mid c_j \geq \gamma^{(r)}B_r^{-1}a_j\gamma^{(r)})$$

$$= \exp\left\{t \sum_{j\in S_r} \gamma_j^{(r)}\hat{x}_j\right\} \prod_{j\notin S_r} \{\exp(\max(0, \gamma^{(r)}B_r^{-1}a_j)t\hat{x}_j)M_j(\hat{x}_jt)\}$$

$$\geq \exp\left\{t \sum_{j\in S_r} \gamma_j^{(r)}\hat{x}_j + \sum_{j\notin S_r} t\hat{x}_j(\gamma^{(r)}B_r^{-1}a_j)\right\} \prod_{j\notin S_r} M_j(\hat{x}_jt)$$

$$= \exp(t\gamma^{(r)}B_r^{-1}b) \prod_{j\notin S_r} M_j(\hat{x}_jt)$$

$$= E(\exp(tz^*) \mid E_r, \gamma^{(r)}) \prod_{j\notin S_r} M_j(\hat{x}_jt).$$

Hence

$$E\left(\exp\left(t \sum_{j=1}^{n} c_j\hat{x}_j\right) \Big| E_r\right) \geq E\left(\exp(tz^*) \prod_{j\notin S_r} M_j(\hat{x}_jt) \Big| E_r\right)$$

and so

$$\prod_{j=1}^{n} M_j(\hat{x}_jt) \geq E\left(\exp(tz^*) \prod_{j\notin \mathrm{OPT}} M_j(\hat{x}_jt)\right)$$

where OPT is the (random) set of optimal basic variables.

Hence

$$E\left(\exp(tz^*) \Big/ \prod_{j\in \mathrm{OPT}} M_j(\hat{x}_jt)\right) \leq 1$$

and so

$$M_{z^*}(t) \leq \max_r \prod_{j\in S_r} M_j(\hat{x}_jt) \tag{2.11}$$

and the theorem follows.

Corollary 2.5. *Suppose that $c_1, c_2, \ldots, c_n$ are independent exponential random variables each with parameter λ and that $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$ is a feasible solution to* (1.1) *with $\hat{x} = \max\{\hat{x}_j : j = 1, 2, \ldots, n\}$. Then*

$$\Pr(z^* \geq (1+\delta)m\lambda^{-1}x) \leq \exp(-m(\delta - \log(1+\delta))) \quad \text{for } \delta > 0. \tag{2.12}$$

Proof. Put $a = (1+\delta)m\lambda^{-1}\hat{x}$ and $t = \delta\lambda/((1+\delta)\hat{x})$ in (2.10) and use Theorem 2.4.

Corollary 2.6. *Suppose that $c_1, c_2, \ldots, c_n$ are independent uniform* [0, 1] *random variables and $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$ is a feasible solution to* (1.1) *with $\hat{x} = \max\{\hat{x}_j : j = 1, 2, \ldots, n\}$. Then*

$$\Pr(z^* \geq (1+\delta)m\hat{x}) \leq \exp(-m(\delta - \log(1+\delta))) \quad \text{for } \delta > 0. \tag{2.13}$$

Proof. A uniform [0, 1] random variable X is smaller in distribution than an exponentially distributed random variable Y with parameter 1, for, if $t > 0$,

$$\Pr(Y > t) = e^{-t} \geq \max(0, 1-t) = \Pr(X > t).$$

Thus the result of Corollary 2.5 applies with $\lambda = 1$.

3. Solving quadratic assignment problems

We now consider the computational consequences of our results for the exact solution of the Quadratic Assignment Problem (QAP):

$$\begin{aligned} &\text{minimise} \quad \zeta = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} a_{ijkl}x_{ij}x_{kl} \\ &\text{subject to} \quad \sum_{i=1}^{n} x_{ij} = 1, \; j = 1, 2, \ldots, n, \\ &\qquad\qquad \sum_{j=1}^{n} x_{ij} = 1, \; i = 1, 2, \ldots, n, \\ &\qquad\qquad x_{ij} = 0 \text{ or } 1. \end{aligned} \tag{3.1}$$

This problem has a large number of important applications including the location of rooms in buildings, the location of symbol keys on type-writers, the location of files on disks and the design of printed circuits - see Burkard [1] for a recent review.

The problem is known to be NP-Hard (Garey and Johnson, [5]) and extremely difficult to solve exactly. Problems of size $n = 20$ cannot be solved on a routine basis.

We consider the case where the a_{ijkl}'s are independent uniform [0, 1] random variates. Now it is easy to show, Burkard and Fincke [2], that

$$\Pr(\zeta_{\max} \geq (1+o(1))\zeta_{\min}) = o(1) \tag{3.2}$$

for suitable o(1) terms. Here $\zeta_{\min}$ (resp. $\zeta_{\max}$) denotes the minimum (resp. maximum) value of ζ in the QAP (3.1). Thus any algorithm almost always finds a good solution.

On the other hand, Frieze and Yadegar [4] showed that many proposed branch and bound algorithms use bounds which are weaker than the bound derived from the following LP relaxation of QAP:

$$\text{minimise} \quad z=\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} a_{ijkl}\, y_{ijkl}$$

$$\begin{aligned}
\text{subject to} \quad & \sum_{i=1}^{n} x_{ij}=1, && j=1,2,\ldots,n,\\
& \sum_{j=1}^{n} x_{ij}=1, && i=1,2,\ldots,n,\\
& \sum_{i=1}^{n} y_{ijkl}=x_{kl}, && j,k,l=1,2,\ldots,n,\\
& \sum_{j=1}^{n} y_{ijkl}=x_{kl}, && i,k,l=1,2,\ldots,n,\\
& \sum_{k=1}^{n} y_{ijkl}=x_{ij}, && i,j,l=1,2,\ldots,n,\\
& \sum_{l=1}^{n} y_{ijkl}=x_{ij}, && i,j,k=1,2,\ldots,n,\\
& y_{ijij}=x_{ij}, && i,j=1,2,\ldots,n,\\
& x_{ij}\geq 0, && i,j=1,2,\ldots,n,\\
& y_{ijkl}\geq 0, && i,j,k,l=1,2,\ldots,n.
\end{aligned} \tag{3.3}$$

Let now z^* denote the minimum value of z in the LP relaxation (3.3). It follows from Theorem 2.1 that

$$E(z^*)\leq 5n+O(1) \tag{3.4}$$

(Fix $x_{ij}=y_{ijij}=1/n$ for $i,j=1,2,\ldots,n$ and then consider the restricted linear program remaining, with feasible solution

$$y_{ijkl}=\begin{cases}1/[n(n-1)] & \text{if } i\neq k, j\neq l,\\ 0 & \text{otherwise.)}\end{cases}$$

On the other hand as in (3.2) we have

$$\Pr(\zeta_{\min}\leq(1-o(1))n^2/2)=o(1). \tag{3.5}$$

Thus for large n, the expected duality gap is enormous. As a consequence, we can use the results of Section 2 to show the following result, which we feel gives some insight into the difficulty of the QAP.

Theorem 3.1. *Consider any branch-and-bound algorithm for solving the QAP (3.1) that branches by setting variables x_{ij} to 0 or 1 and bounds by using the LP(3.3). The number of branch nodes explored is at least $n^{(1-o(1))n/4}$ with probability $1-o(1)$ as $n\to\infty$.*

Proof. A *probe* for the QAP problem QAP_n in (3.1) is a pair (A,B) of disjoint subsets of N^2, where $N=\{1,2,\ldots,n\}$. The value $v_n(A,B)$ is the minimum value of z in the LP relaxation (3.3) subject to the further constraints

$$x_{ij}=\begin{cases}1 & \text{for all } (i,j)\in A,\\ 0 & \text{for all } (i,j)\in B.\end{cases}$$

The probe (A,B) is *feasible* if this program, say $LP_n(A,B)$, is feasible.

Clearly each node in a branch-and-bound tree for QAP_n corresponds to a probe, and the lower bound at a node is the value of the probe. Note that if some variable x_{ij} has a fixed value in all integer feasible solutions to $LP_n(A,B)$ then it has this fixed value in *all* feasible solutions. Hence from any branch-and-bound tree we may construct one which is no larger and in which each node corresponds to a feasible probe. Thus we can restrict attention to such trees.

Let the random variable ζ_n^* be the optimum value of QAP_n. Given non-negative integer-valued functions $\alpha=\alpha(n)$ and $\beta=\beta(n)$ define the random variable $v_n^*(\alpha,\beta)$ to be the maximum value of $v_n(A,B)$ over all feasible probes (A,B) with $|A|\leq\alpha(n)$ and $|B|\leq\beta(n)$. If $v_n^*(\alpha,\beta)<\zeta_n^*$ then the number of nodes in any branch-and-bound tree T is at least

$$\binom{\alpha+\beta}{\alpha}\geq(\beta/\alpha)^{\alpha}.$$

To see this, let branching to the left mean setting some variable $x_{ij}=0$ and branching to the right mean setting $x_{ij}=1$. If $v_n^*(\alpha,\beta)<\zeta_n^*$ then the tree T must contain all possible paths from the root with α right turns and β left turns.

Let $\varepsilon=\varepsilon(n)>0$ and $\varepsilon(n)\to 0$ slowly as $n\to\infty$, say $\varepsilon(n)\sim(\log n)^{-1}$. Let $\alpha=\alpha(n)$ and $\beta=\beta(n)$ be non-negative integer-valued functions such that $\alpha(n)\sim(1-\varepsilon)n$ and $\beta(n)\sim n^{5/4}(\log n)^{-3}$. We shall show that, with probability $\to 1$ as $n\to\infty$,

$$v_n^*(\alpha,\beta)<\zeta_n^* \tag{3.6}$$

and thus the number of nodes in each branch-and-bound tree for QAP_n is at least

$$(\beta/\alpha)^{\alpha}=n^{(1-o(1))n/4}.$$

In order to prove (3.6) we shall consider feasible probes of the form (A,ϕ). Let $m=m(n)$ be a positive integer-valued function such that

$$m(n)\sim 3(n^3\log n)^{1/4}.$$

We shall show that for n sufficiently large

$$v_n^*(\alpha,\beta)\leq v_n^*(n-m,0), \tag{3.7}$$

and

$$\Pr\{v_n^*(n-m,0)<\zeta_n^*\}\to 1 \quad \text{as } n\to\infty. \tag{3.8}$$

This will of course establish (3.6) and complete our proof.

We can dispose of (3.7) quickly, by postponing the (deterministic) work to a later lemma. Consider any feasible probe (A, B) with $|A| \leq \alpha(n)$ and $|B| \leq \beta(n)$. By Lemma 3.2 below, if n is sufficiently large there is a feasible proble $(\bar{A}, B)$ such that $A \subseteq \bar{A}$, $|\bar{A}| = n - m$ and $\bar{A}$ 'covers' B, that is for each $(i, j) \in B$ either some $(i, k) \in \bar{A}$ or some $(k, j) \in \bar{A}$. Then

$$v_n(A, B) \leq v_n(\bar{A}, B) = v_n(\bar{A}, \phi) \leq v_n^*(n-m, 0),$$

which establishes (3.7).

It remains for us here to prove (3.8). Let (A, ϕ) be a feasible probe with $|A| = n - m$. Our first step is to define a feasible solution to $\mathrm{LP}_n(A, \phi)$. Let

$$X = \{i \in N: (i, j) \in A \text{ for some } j\}, \quad \bar{X} = N \backslash X,$$

$$Y = \{j \in N: (i, j) \in A \text{ for some } i\}, \quad \bar{Y} = N \backslash Y.$$

Of course $|\bar{X}| = |\bar{Y}| = m$. Let $a(A)$ be the subarray of $a = (a_{ijkl})$ with $i, k \in \bar{X}$ and $j, l \in \bar{Y}$.

Consider the restricted linear program $\mathrm{RLP}_n(A)$ defined as follows. Start with the LP relaxation (3.3) of the QAP corresponding to the subarray $a(A)$. Now fix

$$x_{ij} = y_{ijij} = 1/m$$

for all $i \in \bar{X}$ and $j \in \bar{Y}$, thus leaving $m^4 - m^2$ variables y_{ijkl} where $i, k \in \bar{X}$ and $j, l \in \bar{Y}$ and $(i, j) \neq (k, l)$. Let y^* be an optimal solution to $\mathrm{RLP}_n(A)$, and let the random variable r_n be the corresponding minimum value of the objective function.

Now let

$$\hat{x}_{ij} = \begin{cases} 1 & \text{if } (i, j) \in A, \\ 1/m & \text{if } i \in \bar{X}, j \in \bar{Y}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\hat{y}_{ijkl} = \begin{cases} 1 & \text{if } (i, j) \in A \text{ and } (k, l) \in A, \\ 1/m & \text{if } (i, j) \in A \text{ and } k \in \bar{X},\ l \in \bar{Y}, \\ & \text{of if } i \in \bar{X}, j \in \bar{Y} \text{ and } (k, l) \in A, \\ y^*_{ijkl} & \text{if } i, k \in \bar{X} \text{ and } j, l \in \bar{Y}, \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to check that this does define a feasible solution $\hat{x}$, $\hat{y}$ to $\mathrm{LP}_n(A, \phi)$. The corresponding random value $\hat{z}_n(A)$ of the objective function is given by

$$\hat{z}_n(A) = \sum_{(i,j)\in A} \sum_{(k,l)\in A} a_{ijkl} + \frac{1}{m} \sum_{(i,j)\in A} \sum_{k \in \bar{X}} \sum_{l \in \bar{Y}} a_{ijkl} + \frac{1}{m} \sum_{i \in \bar{X}} \sum_{j \in \bar{Y}} \sum_{(k,l)\in A} a_{ijkl} + r_n. \tag{3.9}$$

Let the random variable $\hat{z}_n^*$ be the maximum value of $\hat{z}_n(A)$ over all feasible probes (A, ϕ) with $|A| = n - m$. Of course $v_n^*(n-m, 0) \leq \hat{z}_n^*$. We shall show that

$$\Pr\{\hat{z}_n^* < \zeta_n^*\} \to 1 \quad \text{as } n \to \infty, \tag{3.10}$$

which will then complete our proof.

We shall use an inequality of Hoeffding. Let $X_1, \ldots, X_k$ be independent random variables each uniformly distributed on $[0, 1]$. Then by a special case of theorem 1 of [6], if $0 < \varepsilon < 1$,

$$\Pr\{\Sigma X_j \leq (1-\varepsilon)k/2\} = \Pr\{\Sigma X_j \geq (1+\varepsilon)k/2\} \leq \exp(-\tfrac{1}{2}\varepsilon^2 k). \tag{3.11}$$

Let $\delta = \delta(n) = (\log n/n)^{1/2}$. Now ζ_n^* is the minimum over all $n!$ possible assignments of a sum of n^2 costs. Hence by (3.11)

$$\Pr\{\zeta_n^* \leq (\tfrac{1}{2} - \delta)n^2\} \leq n! \exp\{-2\delta^2 n^2\} < \exp(n \log n - 2n \log n) \to 0 \quad \text{as } n \to \infty.$$

Let $\Sigma_{(1)}, \Sigma_{(2)}, \Sigma_{(3)}$ be the first three terms in the expression (3.9) for $\hat{z}_n(A)$. Now $\Sigma_{(1)}$ is a sum of $(n-m)^2$ costs and so, by (3.11),

$$\Pr\{\Sigma_{(1)} \geq (\tfrac{1}{2} + \delta)(n-m)^2\} \leq \exp(-2\delta^2(n-m)^2) = \exp(-(2+o(1))n \log n).$$

Similarly $m(\Sigma_{(2)} + \Sigma_{(3)})$ is a sum of $2m^2(n-m)$ costs, and so

$$\Pr\{\Sigma_{(2)} + \Sigma_{(3)} \geq (\tfrac{1}{2} + \delta)2m(n-m)\} \leq \exp(-4\delta^2 m^2(n-m))$$
$$= \exp(-(36+o(1))(n \log n)^{3/2}).$$

Now let Σ_n^* be the maximum value of $\Sigma_{(1)} + \Sigma_{(2)} + \Sigma_{(3)}$ over all feasible probes (A, ϕ) with $|A| = n - m$. Then, by the above,

$$\Pr\{\Sigma_n^* \geq (\tfrac{1}{2} + \delta)(n^2 - m^2)\} \leq \binom{n^2}{n-m} \exp(-(2+o(1))n \log n)$$
$$\leq \exp(n \log(en) - (2+o(1))n \log n) \to 0 \quad \text{as } n \to \infty.$$

Next consider the random value $r_n = r_n(A)$ of $\mathrm{RLP}_n(A)$. The linear program $\mathrm{RLP}_n(A)$ has $m^4 - m^2$ variables, $4m^3$ constraints, a constant term m in the objective function, and a feasible solution

$$y_{ijkl} = \begin{cases} (m(m-1))^{-1} & \text{if } i, k \in \bar{X},\ i \neq k \text{ and } j, l \in \bar{Y},\ j \neq l, \\ 0 & \text{otherwise.} \end{cases}$$

Let $s(n) = m + 2(4m^3/m(m-1))$. Then by corollary (2.6) with $\delta = 1$,

$$\Pr\{r_n \geq s(n)\} \leq \exp(-(1 - \log 2)4m^3).$$

Let the random variable r_n^* be the maximum value of $r_n(A)$ over all feasible probes (A, ϕ) with $|A| = n - m$. Then

$$\Pr\{r_n^* \geq s(n)\} \leq \binom{n}{m}^2 \exp(-(1 - \log 2)4m^3) \to 0 \quad \text{as } n \to \infty.$$

We may at last complete the proof of (3.10). For with probability → 1 as $n \to \infty$,

$$\hat{z}_n^* \leq \Sigma_n^* + r_n^* < (\tfrac{1}{2}+\delta)(n^2 - m^2) + s(n) < (\tfrac{1}{2}-\delta)n^2 < \zeta_n^*,$$

as required.

Lemma 3.2. *Let $G = (U, V, E)$ be a bipartite graph with $|U| = |V| = n$ and $|E| \geq n^2 - \beta$, where $0 < \beta \leq n^2/17$ and $n \geq 141$. Suppose that G has a perfect matching. Then there exist $U' \subseteq U$, $V' \subseteq V$ such that*

(i) *$G(U', V')$ is complete,*

(ii) *$G(U \setminus U', V \setminus V')$ has a perfect matching,*

(iii) *$|U'| = |V'| = \lfloor (n/2)/\lceil 4\beta/n \rceil \rfloor$.*

Here $G(U', V')$ denotes the induced subgraph of G with vertex set $U' \cup V'$.

Proof. We may assume that $\beta \geq n/2$. Let $r = \lceil 4\beta/n \rceil$. The number of vertices in U of degree $\leq n - r$ is at most $\beta/r \leq n/4$; and similarly for V. Since G has a perfect matching we may pick $U_1 \subseteq U$, $V_1 \subseteq V$ such that $|U_1| = |V_1| = n_1 \geq n/2$, $G(U \setminus U_1, V \setminus V_1)$ has a perfect matching, and in $G(U_1, V_1)$ all vertex degrees are at least $n_1 - r + 1$. By adding vertices alternately from U_1 and V_1 we may find $U_2 \subseteq U_1$, $V_2 \subseteq V_1$ such that $|U_2| = |V_2| = \lfloor n_1/r \rfloor$ and the graph $G(U_2, V_2)$ is complete.

Now let $U_3 = U_1 \setminus U_2$, $V_3 = V_1 \setminus V_2$ and consider the graph $G_3 = G(U_3, V_3)$. Note that $|U_3| = |V_3| = n_1 - \lfloor n_1/r \rfloor = n_3$ say. It will be sufficient for us to show that G_3 has a perfect matching. This will be true if each vertex degree is at least $n_3/2$ (by for example Hall's theorem). But each vertex degree is at least $n_3 - r$ and so it remains for us to show that $n_3 - 2r \geq 0$. However, $2 \leq r \leq 4n/17 + 1$, and so

$$n_3 - 2r \geq n/2 - (n/2r + 2r) \geq n/2 - \max\{n/4 + 4,\ 8n/17 + \tfrac{33}{8}\} \geq 0 \quad \text{since } n \geq 141.$$

For further related work see [8].

References

[1] R.E. Burkard, "Locations with spatial interactions—Quadratic assignment problem", in: R.L. Francis and Mirchandani, eds., *Discrete location theory* (Academic Press, New York, 1984).

[2] R.E. Burkard and U. Fincke, "The asymptotic probabilistic behaviour of quadratic sum assignment problems", *Zeitschrift für Operations Research* 27 (1983) 73-81.

[3] A.M. Frieze, "Complexity of a 3-dimensional assignment problem", *European Journal of Operational Research* 13 (1983) 161-164.

[4] A.M. Frieze and J. Yadegar, "On the quadratic assignment problem", *Discrete Applied Mathematics* 5 (1983) 89-98.

[5] M.R. Garey and D. S. Johnson, *Computers and intractability* (Freeman, San Francisco, 1979).

[6] W. Hoeffding, "Probability inequalities for sums of bounded random variables", *Journal of the American Statistical Association* 58 (1963) 13-30.

[7] R.M. Karp, Lecture at the NIHE Summer School on Combinatorial Optimisation, Dublin, 1983.

[8] C.J.H. McDiarmid, "On the greedy algorithm with random costs", to appear.

[9] D.W. Walkup, "On the expected value of a random assignment problem", *SIAM Journal on Computing* 8 (1979) 440-442.

Mathematical Programming 35 (1986) 17-31
North-Holland

A FINITE CHARACTERIZATION OF K-MATRICES IN DIMENSIONS LESS THAN FOUR

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The class of real $n \times n$ matrices M, known as K-matrices, for which the linear complementarity problem $w - Mz = q$, $w \geq 0$, $z \geq 0$, $w^T z = 0$ has a solution whenever $w - Mz = q$, $w \geq 0$, $z \geq 0$ has a solution is characterized for dimensions $n < 4$. The characterization is finite and 'practical'. Several necessary conditions, sufficient conditions, and counterexamples pertaining to K-matrices are also given. A finite characterization of completely K-matrices (K-matrices all of whose principal submatrices are also K-matrices) is proved for dimensions <4.

Key words: Linear complementarity problem, K-matrix, Q_0-matrix, finite characterization, Q-matrix.

1. Introduction

Let E^n be the n-dimensional Euclidean space and let $E^{n \times n}$ be the set of real $n \times n$ matrices. For $M \in E^{n \times n}$, M_{ij} denotes the (i,j) entry in M, and for I, $J \subset \{1, \ldots, n\}$, $M_{I.}$ is the submatrix of M consisting of the rows indexed by I; and $M_{.J}$ consists of the columns indexed by J. The jth column of M is denoted either M_j or $M_{.j}$, the ith row of M is denoted by $M_{i.}$.

Given a matrix $M \in E^{n \times n}$ and vector $q \in E^n$, the linear complementarity problem, denoted by (q, M), is to find vectors w, $z \in E^n$ such that

$$w - Mz = q, \qquad w \geq 0, \quad z \geq 0, \quad w^T z = 0. \tag{LCP}$$

This problem arises in such diverse areas as economics, game theory, linear programming, mechanics, lubrication, numerical analysis, and nonlinear optimization. Generally in a particular application area the matrix M has a special structure (e.g.,

* Partially supported by NSF Grant MCS-8207217.
** Research partially supported by NSF Grant No. ECS-8401081.