On the Independence and Chromatic Numbers of Random Regular Graphs

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Let \( G_r \) denote a random \( r \)-regular graph with vertex set \{1, 2, ..., \( n \)\} and \( \alpha(G_r) \) and \( \chi(G_r) \) denote respectively its independence and chromatic numbers. We show that with probability going to 1 as \( n \to \infty \) respectively

\[
\left| \alpha(G_r) - \frac{2n}{r} \left( \log r - \log \log r + 1 - \log 2 \right) \right| \leq \frac{en}{r}
\]

and

\[
\left| \chi(G_r) - \frac{r}{2} \left( \log r - \frac{8r \log \log r}{(\log r)^2} \right) \right| \leq \frac{8r \log \log r}{(\log r)^2}
\]

provided \( r = o(n) \), \( 0 < 1/3 \), \( 0 < \varepsilon < 1 \), are constants, and \( r \geq r_\varepsilon \), where \( r_\varepsilon \) depends on \( \varepsilon \) only. © 1992 Academic Press, Inc.

This paper is concerned with the independence and chromatic numbers of random regular graphs. Thus let \( \text{REG}(n, r) \) denote the set of \( r \)-regular graphs with vertex set \([n] = \{1, 2, ..., n\}\}. Let \( G_r \) denote a random graph sampled uniformly from \( \text{REG}(n, r) \). We use \( \alpha \) and \( \chi \) for independence and chromatic numbers, respectively.

In random graph theory, these have been studied by, inter alia, Matula [13], Grimmett and McDiarmid [9], Bollobás and Erdős [6], Shamir and Spencer [14], Bollobás [5], Frieze [8], and Łuczak [12]. The aim of this paper is to extend the results of [8, 12] to \( G_r \) and prove
THEOREM. (a) Let $0 < \varepsilon < 1$ be fixed. There exists a constant $r_\varepsilon$ such that if $r \geq r_\varepsilon$, $r = o(n^{1/3})$, $\theta < 1/3$ constant, then

$$\left| \chi(G_r) - \frac{2n}{r} \left( \log r - \log \log r + 1 - \log 2 \right) \right| \leq \frac{\varepsilon n}{r}$$

with probability going to 1 as $n \to \infty$.

(b) Moreover, for some constant $r_0$ and $r_0 < r = o(n^\theta)$, we have

$$\frac{r}{2 \log r} \leq \chi(G_r) \leq \frac{r}{2 \log r} \left( 1 + \frac{32 \log \log r}{\log r} \right)$$

with probability going to 1 as $n \to \infty$.

(All logarithms are natural.)

Proof of the Theorem. We will first proceed under the assumption that $r$ is constant. The extension to $r$ growing will then be straightforward. We shall use the model of Bender and Canfield [2] and Bollobás [4] to study $G_r$. Specifically, we will adopt the configuration terminology of [4]. We let $W = [rn]$ and $W_i = \{(i-1)r + 1, \ldots, ir\}$, $i = 1, 2, \ldots, n$, be a partition of $W$ into $n$ sets of size $r$. For $w \in W$ we define $\psi(w) = \lfloor w/r \rfloor$ so that $w \in W_{\psi(w)}$ holds.

A configuration on a set $Z$, $|Z| = 2k$, is a partition of $Z$ into $k$ pairs. $\Phi_Z$ denotes the set of configurations on $Z$. If $Z \subseteq W$ and $F \in \Phi_Z$ we let $\mu(F)$ be the multigraph with vertex set $[n]$ and $k$ edges $\{\psi(x), \psi(y) : \{x, y\} \in F\}$.

We consider $\Phi = \Phi_W$ as a probability space in which each $F \in \Phi$ is equally likely. Let $Q$ be a property of the graphs in $\text{REG}(n, r)$ and let $Q^*$ be a property of the configurations in $\Phi$. Suppose these properties are such that for $G_r \in \text{REG}(n, r)$ and $F \in \mu^{-1}(G_r)$, $G_r$ has $Q$ if and only if $F$ has $Q^*$. All we shall need from [2, 4] is

$$\Pr(G_r \in Q) \leq e^{r^2} \Pr(F \in Q^*).$$

(0)

In the analysis we only claim that inequalities hold for $r$ and $n$ sufficiently large and $\varepsilon$ sufficiently small.

It is well known that the binomial random variable $B(n, p)$ is sharply concentrated around its expected value $np$, if this is large.

We will rather loosely refer to the following as the "Chernoff bounds":

$$\Pr(B(n, p) \leq (1 - \beta) np) \leq e^{-\beta^2 np/2}$$

$$\Pr(B(n, p) \geq (1 + \beta) np) \leq e^{-\beta^2 np/3}$$

for $0 \leq \beta \leq 1$. 
We will concentrate first on the independence number. The most difficult task is to bound \( \alpha(F) = \alpha(\mu(F)) \) from below, with high probability. To do this we will generate a random \( F \) in a somewhat complicated way. Our purpose is to use the result of [8] “halfway” through the construction.

For a multigraph \( H \) and vertex \( v \) of \( H \) we let \( d(v, H) \) denote the degree of \( v \) in \( H \), where loops count twice.

**Step 1.** Let \( r = r - \lceil r^{1/2} \log r \rceil \) and \( m_1 = \lfloor r n/2 \rfloor \), and let \( X = (x_1, x_2, \ldots, x_{2m_1}) \) be a random member of \([n]^{2m_1}\); i.e., \( x_1, x_2, \ldots, x_{2m_1} \) are chosen independently at random from \([n]\). Let now \( G_1 \) denote the multigraph with vertex set \([n]\) and edge set \( \{x_{2i-1}x_{2i} : 1 \leq i \leq m_1\} \).

**Step 2.** The next step is to delete edges from \( G_1 \) so that each vertex has degree at most \( r \) and to construct most of \( F \).

Let \( d_j = d(j, G_1) \) and \( Y_1, Y_2, \ldots, Y_n \) be a partition of \( Y = [2m_1] \) with \( |Y_j| = d_j \) for \( 1 \leq j \leq n \). In fact let \( Y_1 = \{d_1\} \) and \( Y_i = \{\sum_{j=1}^{i-1} d_j\} \setminus \{\sum_{j=1}^{i-1} d_j\} \) for \( i \geq 2 \). We construct a configuration \( F_1 \) on \( Y = \bigcup_{j=1}^n Y_j \).

begin
\( F_1 := \emptyset; Y' := Y \) for \( 1 \leq j \leq n; \)
for \( t := 1 \) to \( m_1 \) do
begin
randomly choose \( p_{2t-1} \in Y_{2t-1}; Y'_{2t-1} := Y'_{2t-1} \setminus \{p_{2t-1}\}; \)
randomly choose \( p_{2t} \in Y'_{2t}; Y_{2t} := Y_{2t} \setminus \{p_{2t}\}; \)
\( F_1 := F_1 \cup \{\{p_{2t-1}, p_{2t}\}\} \)
end

**Claim 1.** \( F_1 \) is a random configuration on \( Y \).

**Proof.** \( p_1, p_2, \ldots, p_{2m_1} \) is a random permutation of \( Y \) since interchanging \( p_k, p_{k+1} \) yields the same distribution of permutations. Each partition arises from \( 2^{m_1} \) permutations.

For \( s \in Y \) let its rank \( \rho(s) = s - \sum_{i=1}^{j-1} d_i \), where \( s \in Y_j \), so that \( Y_j \) has elements of rank \( 1, 2, \ldots, d_j \). Let

\[
F'_1 = \{\{p, q\} \in F_1 : \max\{\rho(p), \rho(q)\} \leq r\},
\]

\[
F_2 = \{\{\sigma(p), \sigma(q)\} : \{p, q\} \in F'_1\},
\]

where if \( p \in Y_j \), \( \sigma(p) = (j-1)r + \rho(p) \), and \( Z = \bigcup_{e \in F_2} e \).

**Claim 2.** \( F_2 \) is a random configuration on \( Z \).

**Proof.** \( F_1 \) is a random configuration on \( Y \) implies \( F'_1 \) is a random configuration on \( \bigcup_{e \in F_1} e \), since any such configuration has the same number
of extensions to an $F_1$. Since $F_2$ is obtained by a fixed relabelling of
$\bigcup_{e \in \mathcal{E}_1} e$, the result follows. □

We let $G_2$ be the multigraph $\mu(F_2)$.

**Step 3.** We now enlarge $F_2$ so that it “covers” the whole of $W$.

Suppose $Z \neq W$, $x_1, x_2 \in W - Z$, and $Z' = Z \cup \{x_1, x_2\}$. We define a
function $f: \Phi_{Z'} \to \Phi_{Z}$ as follows: let $F' \in \Phi_{Z'}$.

(a) If $\{x_1, x_2\} \in F'$ then

$$f(F') = F' - \{\{x_1, x_2\}\},$$

otherwise

(b) suppose $\{x_1, z_1\}, \{x_2, z_2\} \in F', (z_1 \neq z_2)$, then

$$f(F') = (F' \cup \{\{z_1, z_2\}\}) - \{\{x_1, z_1\}, \{x_2, z_2\}\}.$$

**Claim 3.** If $F \in \Phi_Z$ then $|f^{-1}(F)| = |Z| + 1$.

**Proof.** If $F' \in f^{-1}(F)$ then either

(a) $F' = F \cup \{\{x_1, x_2\}\}$ or

(b) $F' = F \cup \{\{x_1, z_1\}, \{x_2, z_2\}\} - \{\{z_1, z_2\}\}$ for some $\{z_1, z_2\} \in F$. □

It follows from Claim 2 and Claim 3 that the following algorithm generates a random configuration $F'_2$ on $Z'$:

**ADD($F_2, x_1, x_2$):**

begin

With probability $(|Z| + 1)^{-1}$ let $F'_2 = F_2 \cup \{\{x_1, x_2\}\}$ else randomly choose

$\{z_1, z_2\} \in F_2$ (randomly ordered $z_1$, $z_2$) and then let

$F'_2 = (F_2 \cup \{\{x_1, z_1\}, \{x_2, z_2\}\}) - \{\{z_1, z_2\}\}$

Output $F'_2$

end

Hence if $W - Z = \{x_1, x_2, \ldots, x_{2s}\}$ the following algorithm constructs a random configuration $F$ on $W$:

**FINISH**

begin

$F := F_2$,

for $i = 1$ to $s$ do $F := ADD(F, x_{2i-1}, x_{2i})$

end
We will now show that with high probability

(i) \( G_1 \), and hence \( G_2 \), has an independent set of the required size.

(ii) Algorithm FINISH does not disturb this set too much.

To prove (i) we observe that if \( G_{n,m} \) denotes the standard random graph with vertex set \([n]\) and \( m \) edges then \( G_{n,m_1} \) can be generated by adding a random number of extra random edges to the graph obtained by deleting loops and coalescing multiplied edges in \( G_1 \).

Now it was shown in [8] that if \( r_1 \geq r_\varepsilon \) then

\[
\Pr \left( \alpha(G_{n,m_1}) \leq \frac{2n}{r_1} \left( \log r_1 - \log \log r_1 + 1 - \log 2 - \varepsilon \right) \right) \leq \exp \left\{ -\frac{An}{r_1 (\log r_1)^2} \right\}
\]

for some "constant" \( A = A(\varepsilon) \).

It follows from this and the fact that \( r_1 = r(1 - O(\log r/r^{1/2})) \) that

\[
\Pr \left( \alpha(G_1) \leq \alpha_\varepsilon = \frac{2n}{r} \left( \log r - \log \log r + 1 - \log 2 - \varepsilon \right) \right) \leq \exp \left\{ -\frac{Bn}{r (\log r)^2} \right\},
\]

(1)

where \( B = B(\varepsilon) \).

We now show that the transition from \( F_2 \) to \( F \) does not create too many edges contained in a given large independent set of \( G_1 \) and \( G_2 \).

Now let \( d'_j = d(j, G_2) \) for \( j \in [n] \) and \( S_0 = \{ j : d'_j \leq r - 3r^{1/2} \log r \} \). Our next task is to prove

\[
\Pr \left( |S_0| \geq \frac{n}{r^{1/6}} \right) \leq e^{-Cn/r^{3/2}}
\]

(2)

for some constant \( C > 0 \).

Now if \( k \in S_0 \) then either

(a) \( k \in S_1 = \{ j : d_j \leq r - 2r^{1/2} \log r \} \) or

(b) \( k \in S_2 = \{ j : d_j - d'_j \geq r^{1/2} \log r \} \).

Now

\[
\Pr(1 \in S_1) = \Pr \left( B \left( 2m_1, \frac{1}{n} \right) \leq r_1 - r^{1/2} \log r \right)
\]

(where \( B(\cdot, \cdot) \) denotes a binomial random variable)

\[
\leq e^{-Cn/r^{3/2}}
\]
from the Chernoff bound for the tails of the binomial. Thus
\[ E(|S_1|) \leq ne^{-(\log r)^{1/3}}. \]

Now the events \( i \in S_1, j \in S_1 \) for \( i \neq j \) are not independent. But on the other hand changing any \( x_i \) can only change \( |S_1| \) by at most one and so, by the Hoeffding–Azuma inequality [1]
\[ \Pr(|S_1| \geq ne^{-(\log r)^{1/3}} + u) \leq \exp \left\{ -\frac{2u^2}{rn} \right\}. \]  
(3)

This inequality implies that if \( \xi = \xi(X) \) is a random variable such that
\[ |\xi(X) - \xi(X')| \leq d \]
whenever \( X \) and \( X' \) differ only in one component, then
\[ \Pr(\xi - E(\xi) \geq u) \leq \exp \left\{ -\frac{2u^2}{2m_1 d^2} \right\} \]
(see, for example, Bollobás [7] or McDiarmid [10]).

To handle \( S_2 \) we define \( \delta_{ij}, i \in [m_1], j \in [n] \) by
\[ \delta_{i,j} = \begin{cases} 1 & \text{if} \quad (a) \quad j \in \{x_{2i-1}, x_{2i}\}, \\ 0 & \text{otherwise}. \end{cases} \]

Then \( j \in S_2 \) implies \( \sum_{i=1}^{m_1} \delta_{i,j} \geq r^{1/2} \log r \). So let \( S'_2 = \{ j : \sum_{i=1}^{m_1} \delta_{ij} \geq r^{1/2} \log r \} \supseteq S_2 \) and observe that \( S'_2 \) depends only on \( X \), unlike \( S_2 \) which depends on \( p_1, \ldots, p_{2m_1} \) as well. Let now \( \delta = \delta_{1,1} \). Then
\[ \Pr(\delta = 1) \leq 2\Pr(x_1 = 1 \text{ and } d_1 > r) + 2\Pr(x_2 = 1 \text{ and } d_1 > r) \]
\[ = \frac{2}{n} \left( \Pr(d_1 > r \mid x_1 = 1) + \Pr(d_1 > r \mid x_1 = 2) \right) \]
\[ \leq \frac{4}{n} \Pr(d_1 > r \mid x_1 = 1) \]
\[ = \frac{4}{n} \Pr \left( B \left( 2m_1 - 1, \frac{1}{n} \right) \geq r \right) \]
\[ \leq \frac{4}{n} e^{-(\log r)^{1/4}} \quad \text{(using the Chernoff bound)}. \]
Hence
\[ E\left( \sum_{i=1}^{m_1} \delta_{i,j} \right) \leq 2re^{-\left(\log r\right)^{2/4}} \quad j \in [n] \]
and so
\[ Pr\left( \sum_{i=1}^{m_1} \delta_{i,j} \geq r^{1/2} \log r \right) \leq 2\left( r^{1/2}/\log r \right) e^{-\left(\log r\right)^{2/4}} \quad j \in [n] \]
and so \( E(|S'_2|) \leq 2n(r^{1/2}/\log r) e^{-\left(\log r\right)^{2/4}}. \)

Now changing any \( x_i \) can change at most 4 \( \delta_{i,j} \)'s and hence \( |S'_2| \) by at most 4 and so by the Hoeffding–Azuma inequality
\[
Pr(|S'_2| \geq 2n(r^{1/2}/\log r) e^{-\left(\log r\right)^{2/4}} + u) \leq \exp\left\{ -\frac{u^2}{8rn} \right\}.
\]  
(4)

Inequality (2) follows from (3) and (4) with \( u = n/3r^{1+\theta} \). So let us now assume that \( \alpha(G_2) > \alpha_c \) (see (1)) and \( |S_0| < n/r^{1+\theta} \). We consider the effect of \( T \).

Let \( T \) be an independent set of vertices of \( G_2 \) of size \( \lceil \alpha_c \rceil \). Assume that \( W - Z = \{ x_1, x_2, \ldots, x_{2\tau} \} \) where \( \psi(x_j) \in T \) iff \( j \in \{ 1, 3, 5, 7, \ldots, 2\tau - 1 \} \). We must estimate the number \( \gamma \) of bad edges which (i) are in \( \mu(F) \) and (ii) are contained in \( T \).

Note that an execution of the statement \( \sigma_i \):
\[ F := \text{ADD}(F, x_{2i-1}, x_{2i}) \]
can only contribute to \( \gamma \) if \( i \leq \tau \) and that
\[
\tau \leq r |S_0| + 3r^{1/2} \log r |T|
\leq r \frac{n}{r^{1+\theta}} + 3r^{1/2} \log r \lceil \alpha_c \rceil
\leq \frac{2n}{r^{\theta}}.
\]

But \( \sigma_i \) creates a bad edge only if \( \{ x_{2i-1}, x_{2i} \} \) is not added to \( F \) and the pair \( \{ z_1, z_2 \} \in F \) satisfies \( \{ \psi(z_1), \psi(z_2) \} \cap T \neq \emptyset \). Hence
\[
Pr(\sigma_i \text{ creates a bad edge}) \leq |\{ z_1, z_2 \} \in F : \{ \psi(z_1), \psi(z_2) \} \cap T \neq \emptyset | / |F_2|
\leq \lceil \alpha_c \rceil r |F_2| \leq 4 \frac{\log r}{r}
\]
regardless of the outcome of the execution of FINISH to this point.
Hence $\gamma$ is majorised by $B(\lceil 2n/r^2 \rceil, 4 \log r/r)$ and so

$$Pr \left( \gamma \geq \frac{16n}{r^1 + \delta} \right) \leq \exp \left\{ -\frac{8n}{3r^1 + \delta} \right\}$$  \hfill (5)

using the Chernoff bound.

Note that $\alpha(\mu(F)) \geq \alpha(G_2) - 2\gamma$.
Thus (1), (2), and (5) (and a surreptitious doubling of $\varepsilon$) imply that

$$Pr(\alpha(F) \leq \alpha_\varepsilon) \leq e^{-Dn/r^1 + 2\delta}$$  \hfill (6)

for some $D = D(\varepsilon)$.

To bound $\alpha(F)$ from above is straightforward.

Let now $l = \lceil \alpha_{-\varepsilon} \rceil$ and $Y$ be the random variable which counts the number of independent sets of $\mu(F)$ of size $l$. Then

$$P(\alpha(F) \geq l) \leq E(Y)$$

$$= \binom{n}{l} \prod_{i=1}^{l-1} \left( 1 - \frac{rl - i}{rn - 2i + 1} \right)$$

$$\leq \binom{n}{l} \prod_{i=1}^{l-1} \left( 1 - \frac{rl - i}{rn} \right)$$

$$\leq 2 \left( \frac{n}{l} \right) \exp \left\{ -\frac{rl^2}{2n} \right\}$$

$$\leq 2 \left( \frac{ne}{l} \exp \left\{ -\frac{rl}{2n} \right\} \right)^l$$

$$\leq 2e^{-dl/2}.$$

Hence, the first part of the theorem for constant $r$ follows from (0), (6), and (7).

Now to the second part of the theorem.

The lower bound is immediate from the first part of the theorem, since $\chi(G_r) \geq n/\alpha(G_r)$. For the upper bound we use the fact that the main result of Łuczak [12] implies that for $r \geq r_0$ (= some sufficiently large constant)

$$Pr \left( \chi(G_1) \geq k_0 = \left[ \frac{r}{2 \log r} \left( 1 + \frac{30 \log \log r}{\log r} \right) \right] \right) = o(1),$$

$G_1$ as in Step 1.

Step 2 can only decrease the chromatic number and (6) shows that
if \( G_1 \) has a \( k_o \)-colouring and we use it for \( \mu(F) \) then with probability \( 1 - o(1) \) \( \mu(F) \) has at most

\[
\frac{16n}{r^{1+\theta}} \cdot \frac{r}{2 \log r} \left( 1 + \frac{30 \log \log r}{\log r} \right) \leq \frac{10n}{r^\theta}
\]

edges with both ends of the same colour. The result (for constant \( r \)) will follow if we show that with probability \( 1 - o(1) \), all subgraphs of \( \mu(F) \) with at most \( s_0 = 20n/r^\theta \) vertices can be (re-)coloured with at most \( l = r \log \log r/(\log r)^2 \) colours. We prove this by showing that any subgraph \( H \) induced by \( s \leq s_0 \) vertices satisfies \( \delta(H) < l \) and this is in turn implied by each such \( H \) having less than \( ls/2 \) edges. This latter statement is easy to prove.

\[
Pr(\exists s \leq s_0 \text{ vertices of } \mu(F) \text{ containing } \geq ls \text{ edges})
\]

\[
\leq \sum_{s = 1}^{s_0} \binom{n}{s} \left( \frac{r^2}{rn - rs} \right)^{ls/2}
\]

\[
\leq \sum_{s = 1}^{s_0} \left( \frac{n^e}{s} \right)^s \left( \frac{s^2 e^{-s}}{ls} \right)^{ls/2} \left( \frac{r}{n-s} \right)^{ls/2}
\]

\[
\leq \sum_{s = 1}^{s_0} \left( \frac{e}{n} \right)^{l(1-2)^{-l}} \frac{3r^{ls/2}}{l} = o(1)
\]

and the whole theorem has been proved, for constant \( r \).

Let us now consider the case \( r \to \infty \) but \( r = o(n^\theta) \). The above proof shows that \( \mu(F) \) for a random \( F \in \Omega_w \) has its independence and chromatic numbers in the right range. We have to show that this implies the same for \( G_r \). We rely on the work of McKay and Wormald [11] for this. They give a procedure DEG which takes as input a random \( F \in \Phi_w \) and tries to construct an \( r \)-regular simple graph by eliminating loops and multiple edges. The elimination of a loop or multiple edge involves the addition of at most 4 new edges. The procedure succeeds with probability \( 1 - o(1) \) and it produces each member of \( \text{REG}(n, r) \) with the same probability. Also, it is easy to see that with probability \( 1 - o(1) \) \( F \) has \( O(r^2) \) loops and multiple edges. Thus we need only show that adding \( O(r^2) \) edges to a typical \( F \) does not change \( \alpha \) or \( \chi \) by much. But now \( r^2 = o(n/r) \) for \( r = o(n^\theta) \), and so part (a) requires no work. For part (b) we need to be convinced that the \( O(r^2) = o(n^{2\theta}) \) added edges are sufficiently random so that they usually induce the union of 4 forests which can be 8-coloured. This can be done fairly straightforwardly but requires a fair amount of detail from [11] which is inappropriate here.
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