Discrete Applied Mathematics 6 (1983) 79–83
North-Holland Publishing Company

NOTE

AN EXTENSION OF CHRISTOFIDES HEURISTIC TO THE k-PERSON TRAVELLING SALESMAN PROBLEM

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Received 12 February 1981
Revised 17 March 1982

Christofides heuristic is extended to the problem of finding a minimum length k-person tour of a complete graph using lengths that satisfy the triangular inequality. An approachable upper bound of $\frac{3}{2}$ is demonstrated for the ratio of heuristic to optimum length solutions.

Introduction

This note considers an approximation algorithm for a k-person version of the travelling salesman problem: Let $G=(V,E)$ be a complete graph containing n vertices $v_1,\dots,v_n$. For a positive integer k a k-tour H is a set of k sub-tours denoted $H_1,\dots,H_k$ where

(1a) H_1 is a simple cycle containing at least 3 edges for $i=1,\dots,k$. [We discuss relaxing the lower bound of 3 edges later on.]

(1b) H_i uses vertex v_1 for $i=1,\dots,k$.

(1c) For each $v\in V\backslash\{v_1\}$ there is a unique sub-tour H_i that passes through v.

We imagine k salesmen each starting at vertex v_1 and visiting sets of vertices which collectively include all the vertices in $V\backslash\{v_1\}$.

Suppose next we are given a non-negative distance function $d:E\to\mathbb{R}$ satisfying

$$d(u,v)+d(v,w)\geq d(u,w) \quad \text{for all } u,v,w\in V \tag{2}$$

For a k-tour H its length $d(H)=\sum_{i=1}^{k} d(H_i)$ where, for $S\subset E$, $d(S)=\sum_{e\in S} d(e)$.

We consider the problem: Find a k-tour H of minimum length.

This problem is NP-hard and we concern ourselves here with the worst-case analysis of an approximation algorithm.

When $k=1$ this is the travelling salesman problem. The polynomial time heuristic with the best known performance guarantee is that described by Christofides [1] – see also Cornuéjols and Nemhauser [2]. Other algorithms are analysed in Frieze [5] and Rosenkrantz, Stearns and Lewis [8].

k-person problems are analysed in Frederickson, Hecht and Kim [3] and

Frederickson [4]. The above problem does not seem to have been analysed from this point of view previously.

An extension of Christofides heuristic

The heuristic we describe is a natural generalisation of Christofides heuristic and has almost the same worst-case performance.

Algorithm

Step 1. Find a spanning tree T_0 of G of minimum length among those that have $2k$ edges incident to vertex v_1. (Glover and Klingman [6] show how to solve this problem in $O|V|^2)$ time.)

Step 2. Identify the set of vertices X_0 which are of odd degree in T. Construct a minimum length perfect matching M_0 in the subgraph of G induced by X_0. ($|X|$ is always even and the matching problem is solvable in $O(|X|^3)$ time - see Lawler [7].)

Step 3. $G_0=(V, T_0 \cup M_0)$ is connected and each node is of even degree ($T_0 \cup M_0$ may include repeated edges and set theoretic notation is not entirely accurate). Construct an eulerian cycle EC_0 of G_0 i.e. a cycle that uses each edge of G_0 exactly once. (EC_0 can be constructed in linear time.)

Step 4. We next reduce EC_0 to a k-tour H_0. let U be the set of nodes adjacent to v_1 in T_0. Suppose that EC_0 follows the node sequence $v_1=w_1, w_2, \ldots, w_m=v_1$. Follow the sequence deleting a node w_i if

(a) $w_i \neq v_1$ and w_i has appeared before,

(b) $w_i \in U$ and $v_1 \notin \{w_{i-1}, w_{i+1}\}$.

At the end of Step 4 the node sequence that is left defines a k-tour H_0. Deletions in (a) ensure that each node $v \neq v_1$ is visited once only and deletions in (b) ensure that each sub-tour includes at least two nodes other than v_1.

Now let H^* be a minimum length k-tour.

Theorem 1. $d(H_0) \leq \frac{3}{2} d(H^*)$.

Proof. First note that

$$d(T_0) \leq d(H^*). \tag{3}$$

This is because by deleting one edge not containing v_1 from each of the sub-tours of H^* we obtain a tree satisfying the degree constraint at v_1.

Next suppose that $H^*=\{(w_1, w_2), (w_2, w_3), \ldots, (w_p, w_{p+1}=v_1)\}$ where $p=n+k-1$ and that $X_0=\{w_{i_1}, \ldots, w_{i_{2q}}\} \not\ni v_1$ where $i_1 < \cdots < i_{2q}$. $H=\{(w_{i_1}, w_{i_2}), \ldots, (w_{i_{2q}}, w_{i_1})\}$. If we define the simple cycle $H=\{(w_{i_1}, w_{i_2}), (w_{i_2}, w_{i_3}), \ldots, (w_{i_{2q}}, w_{i_1})\}$, then (2) implies that $d(H) \leq d(H^*)$ and as H is the union of 2 disjoint perfect matchings we have

$d(M_0) \le \frac{1}{2}d(H) \le \frac{1}{2}d(H^*)$. Now

$$d(H_0) \le d(T_0) + d(M_0) \le d(H^*) + \tfrac{1}{2}d(H^*). \qquad \square$$

Theorem 2. *The upper bound of $\frac{3}{2}$ is approachable.*

Proof. We generalise the examples of Cornuéjols and Nemhauser but only produce non-euclidean problems (see Fig. 1).

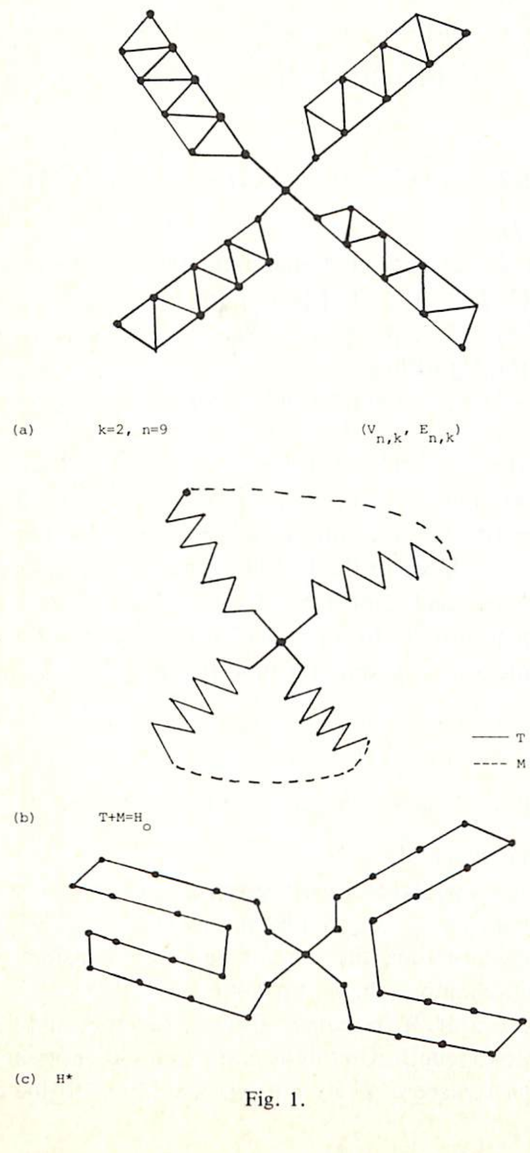

Fig. 1.

Let $G_{n,k}$ be a complete graph with vertices $V_{n,k}=\{0\}\cup\{[m,l]: 1\le m\le n$ and $1\le l\le 2k\}$. Let

$$E_{n,k}=\{(0,[1,l]): 1\le l\le 2k\}\cup\{([t,l],[t+a,l]): 1\le l\le n-a, a=1,2\}.$$

All edges in $E_{n,k}$ are given length 1 and for $(v,w)\notin E_{n,k}$ the length $d(v,w)$ of edge (v,w) is the shortest distance from v to w in the graph $(V_{n,k}E_{n,k})$.

It is straightforward to show that:

(i) The optimal solution H^* is the union of the sub-tours $H_i, i=1,\dots,k$ where for n even:

$$H_i=(0,[1,2i-1],[3,2i-1],\dots,[n-1,2i-1],[n,2i-1],[n-2,2i-1],\dots$$
$$\dots,[2,2i-1],[2,2i],[4,2i],\dots,[n,2i],[n-1,2i],\dots,[1,2i],0),$$

and for n odd

$$H_i=(0,[1,2i-1],[3,2i-1],\dots,[n,2i-1],[n-1,2i-1],\dots \text{etc.})$$

Thus $d(H^*)=2k(n+2)$.

(ii) The minimum degree restricted spanning tree T is the union of the trees $T_i=\{(0,[1,i]),([1,i],[2,i]),\dots,([n-1,i],[n,i])\}$ for $i=1,\dots,2k$.

A minimum matching $M=\{([n,2i-1],[n,2i]): i=1,\dots,k\}$ and $H_0=M\cup T$ is the k-tour produced by the algorithm.

Now $d(H_0)=2k(n+\lfloor\frac{1}{2}n\rfloor+1)$ and so $d(H_0)/d(H^*)\to\frac{3}{2}$ as $n\to\infty$. □

We next consider the restriction (1a) that each H_i has at least 3 edges.

If there is no lower bound to the number of edges, then (2) implies that the optimum solution consists of 1 tour through all the nodes of G plus $k-1$ empty sub-tours, i.e. the problem reduces to the travelling salesman problem.

If the restriction is that each tour has at least 2 edges, i.e. (v_1,v,v_1) is allowable sub-tour then we can transform to a problem where (1a) must hold: replace each node v by a pair of nodes v',v'' and let the modified lengths d' be defined by (where for node v_1 only $v_1'=v_1''$)

$$d'(v',v'')=0,\quad v\in V\setminus\{v_1\},$$

$$d'(v',w')=d'(v',w'')=d'(v'',w')=d'(v'',w'')=d(v,w),\quad v,w\in V,$$

one can easily see that d' satisfies (2).

Problem equivalence is straightforward: given a solution to the original problem replace each sub-tour $(v_1,w_1,\dots,w_p,v_1)$ by $(v_1',w_1',w_1'',\dots,w_p',w_p'',v_1')$.

The lengths of the 2 tours are the same. Conversely consider a solution to the transformed problem. Suppose there is a $v\in V$ such that v',v'' are not adjacent vertices of the same sub-tour. Removing v'' from its sub-tour and putting it adjacent to v' cannot increase total length. Do this as many times as necessary until each v',v'' appear together. Then replace v',v'' by v to get a solution to the original problem of the same length.

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