On the cover time of the emerging giant

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Abstract

Let \( p = \frac{1+\varepsilon}{n} \). It is known that if \( N = \varepsilon^3 n \to \infty \) then with high probability (w.h.p.) \( G_{n,p} \) has a unique giant largest component. We show that if in addition, \( \varepsilon = \varepsilon(n) \to 0 \) then w.h.p. the cover time of \( G_{n,p} \) is asymptotic to \( n \log^2 N \); previously Barlow, Ding, Nachmias and Peres had shown this up to constant multiplicative factors.

1 Introduction

Let \( G = (V, E) \) be a connected graph with vertex set \( V \) of size \( n \) and an edge set \( E \). In a simple random walk \( W \) on a graph \( G \), at each step, a particle moves from its current vertex to a randomly chosen neighbor. For \( v \in V \), let \( C_v \) be the expected time taken for a simple random walk starting at \( v \) to visit every vertex of \( G \). The vertex cover time \( C_G \) of \( G \) is defined as \( C_G = \max_{v \in V} C_v \). The (vertex) cover time of connected graphs has been extensively studied. It was shown by Feige [16], [17], that for any connected graph \( G \), the cover time satisfies

\[
(1 - o(1))n \log n \leq C_G \leq (1 + o(1))\frac{4}{27} n^3.
\]

In a series of papers, Cooper and Frieze have asymptotically established the cover time in a variety of random graph models. The following theorem lists some of the main results. (Here \( A_n \approx B_n \) if \( A_n = (1 + o(1))B_n \) as \( n \to \infty \).)

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Theorem 1. The following asymptotic estimates for the cover time hold with high probability (w.h.p.):

[4] If $G = G_{n,p}$ with $p = \frac{c \log n}{n}, c > 1$, then $C_G \approx \phi(c)n \log n$ where $\phi(c) = c \log \left( \frac{c-1}{c-1} \right)$.

[5] If $G = G_{n,r}$ with $r = O(1)$, a random $r$-regular graph, then $C_G \approx \frac{r-1}{r-2}n \log n$.

[6] Let $G = G_{n,d,r}$ with $d \geq 3$ and $r = \left( \frac{c \log n}{\Upsilon_d} \right)^{1/d}$ be the random geometric graph on $n$ vertices in dimension $d^1$. Then $C_G \approx \phi(c)n \log n$.

[7] If $D = D_{n,p}$ (the random digraph counterpart of $G_{n,p}$), then $C_D \approx \phi(c)n \log n$.

Cooper and Frieze [8] also established the cover time of the giant component $C_1$ of the random graph $G_{n,p}$ with $p = c/n$, where $c > 1$ is a constant. They showed in this setting that w.h.p. the cover time $C_{C_1}$ satisfies

$$C_{C_1} \approx \frac{cx(2-x)}{4(cx - \ln c)} n(\ln n)^2,$$

where $x$ denotes the solution in $(0,1)$ of $x = 1 - e^{-cx}$.

This raises the question as to what happens in $G_{n,p}$ if $p = (1 + \varepsilon)/n$, $\varepsilon > 0$ and we allow $\varepsilon \to 0$. It is known that a unique giant component emerges w.h.p. only when $\varepsilon^3 n \to \infty$. Barlow, Ding, Nachmias and Peres [2] showed that w.h.p.

$$C_{C_1} = \Theta(n \log^2(\varepsilon^3 n)).$$

(1)

Cooper, Frieze and Lubetzky [9] showed that if $C^{(2)}$ denotes the 2-core of the giant component $C_1$ of $G_{n,p}$ ($C_1$ stripped of its attached trees), then, in this range of $p$, w.h.p. $C_{C^{(2)}} \approx \frac{1}{4} \varepsilon n \log^2(\varepsilon^3 n)$, but they were not able to determine the cover time of the giant $C_1$ asymptotically. We do this in the current paper, confirming their conjecture.

We prove the following theorem:

Theorem 2. Let $p = \frac{1 + \varepsilon}{n}$ with $\varepsilon = \varepsilon(n) > 0$, $\varepsilon \to 0$ such that $\varepsilon^3 n \to \infty$. Let $C_1$ be the giant component of $G_{n,p}$. Then w.h.p.

$$C_{C_1} \approx n \log^2(\varepsilon^3 n).$$

Our proof is very different from the proof in [9]. We will use the notion of a Gaussian Free Field (GFF). This was used in the breakthrough paper of Ding, Lee and Peres [13] that describes

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1Here $\Upsilon_d$ is the volume of the Euclidean ball of radius one in $\mathbb{R}^d$. The random geometric graph $G = G_{n,d,r}$ is defined as follows: we choose $n$ points independently uniformly at random from $[0,1]^d$ to be the vertices of $G$ and two points are joined by an edge if and only if they are at most distance $r$-apart.
a deterministic algorithm for approximating $C_G$ to within a constant factor. This was later refined by Ding [14] and by Zhai [22]. It is the latter paper that we will use. In the next section, we will describe the tools needed for our proof. Then in Section 3 we will use these tools to prove Theorem 2.

2 Tools

2.1 Gaussian Free Field

Definition 1. For our purposes, given a graph $G = (V,E)$, a GFF is a random vector $(\eta_v, v \in V)$ whose joint distribution is Gaussian with

(i) $E(\eta_v) = 0$ for all $v \in V$.

(ii) $\eta_{v_0} = 0$ for some fixed vertex $v_0 \in V$.

(iii) $E((\eta_v - \eta_w)^2) = R_{\text{eff}}(v,w)$ for all $v, w \in V$.

Note that in particular, $\text{Var}(\eta_v) = E(\eta_v^2) = R_{\text{eff}}(v,v_0)$. (Here $R_{\text{eff}}$ is the effective resistance between $v$ and $w$, when $G$ is treated as an electrical network where each edge is a resistor of resistance one. See Doyle and Snell [15] or Lewin, Peres and Wilmer [20] for nice discussions of this notion). As its name suggests, $R_{\text{eff}}$ is most naturally defined in terms of electrical networks. For us the following mathematical definition will suffice: for a graph $G = (V,E)$ and vertices $v, w \in V$, we use the commute time identity to define

\[
R_{\text{eff}}(v,w) = \frac{\tau(v,w) + \tau(w,v)}{2|E|},
\]

where $\tau(v,w)$ is the expected time for a simple random walk starting at $v$ to reach $w$.

Note that, as suggested by the electrical analog, we have

\[
R_{\text{eff}}(v,w) \leq \text{dist}(v,w).
\]

This is a simple consequence of Rayleigh’s Monotonicity Law (delete all edges except for a shortest path from $v$ to $w$), see [15].

In the continuous setting, the Gaussian free field generalizes Brownian motion (or the Brownian bridge) and can be seen as a model of a random surface. In the discrete setting, the Gaussian Free Field can be seen as generalizing Brownian motion on a line to an analog of Brownian motion on the topology of the graph. In particular, if $G$ is a path with $t$ edges, and
the fixed vertex \( v_0 \) is an endpoint of the path, then the normals \( \eta_v \) in the GFF for the path can be generated in terms of Brownian motion \( W(t) \), by setting \( \eta_v \) to be \( W(\text{dist}(v, v_0)) \).

The important thing for the present paper is a remarkable connection between the Gaussian Free Field on a graph and its cover time. Let us define

\[
M = \mathbf{E}(\max_{v \in V} \eta_v).
\]

Ding, Lee and Peres [13] proved that there are universal constants \( c_1, c_2 \) such that

\[
c_1 |E| M^2 \leq C_G \leq c_2 |E| M^2.
\]

Next let \( R = \max_{v, w \in V} R_{\text{eff}}(v, w) \). Zhai [22] proved the following theorem:

**Theorem 3 (Zhai).** Let \( G = (V, E) \) be a finite undirected graph with a specified vertex \( v_0 \in V \). There are universal positive constants \( c_1, c_2 \) such that if we let \( \tau_{\text{cov}} \) be the first time that all the vertices in \( V \) have been visited at least once for the walk on \( G \) started at \( v_0 \), we have

\[
\Pr \left( \left| \tau_{\text{cov}} - |E| M^2 \right| \geq |E| (\sqrt{\lambda R} \cdot M + \lambda R) \right) \leq c_1 e^{-c_2 \lambda}
\]

for any \( \lambda \geq c_1 \).

Setting \( X = \frac{\tau_{\text{cov}}}{|E| M^2} \), this gives after crude estimates

\[
|EX - 1| \leq \mathbf{E}|X - 1| = \int_0^\infty \Pr(|X - 1| > t) dt \leq C \left( \sqrt{\frac{R}{M^2} + \frac{R}{M^2}} \right)
\]

for a universal constant \( C \). Note that \( R \) and \( M \) do not depend on \( v_0 \) (for \( M \), observe that for any fixed vertex \( w \), \( \mathbf{E}[\max_{v \in V} \eta_v] = \mathbf{E}[\max_{v \in V} (\eta_v - \eta_w)] + \eta_w] = \mathbf{E}[\max_{v \in V} (\eta_v - \eta_w)] \), since the Gaussians have mean 0, see also Remark 1.3 in [22]). After taking the maximum over \( v_0 \) we thus get that \( C_G = \max_{v_0} \mathbf{E} \tau_{\text{cov}} \) satisfies

\[
C_G = |E| M^2 \left( 1 + O \left( \sqrt{\frac{R}{M^2} + \frac{R}{M^2}} \right) \right).
\]

Now, as we will see in the next section, the number of edges in the emerging giant is given by the following theorem:

**Theorem 4.** Let \( G = G_{n, p} \) be as in Theorem 2. Then

\[
|E(C_1)| \approx 2\varepsilon n \quad \text{w.h.p.}
\]

This follows from the work in [11] as we will see in Section 2.2.

Our main contribution is the following theorem:
Theorem 5. Let $G = G_{n,p}$ be as in Theorem 2 and let $M$ the the expected maximum of a GFF on $G$ as defined above. Then

$$M \approx \frac{\log(\varepsilon^3 n)}{(2\varepsilon)^{1/2}} \text{ w.h.p.} \quad (8)$$

This immediately implies Theorem 2 as follows:

Proof of Theorem 2. In view of (6) obtained from Theorem 3, Theorem 5 implies Theorem 2 if we can show that w.h.p. $R = o(M^2)$. Now, we know from (1), (4) and (7) (or from Theorem 5) that w.h.p. $M = \Omega(\varepsilon^{-1/2} \log(\varepsilon^3 n))$. Therefore to prove that $R = o(M^2)$ it will be sufficient to prove

$$R = O \left( \frac{\log(\varepsilon^3 n)}{\varepsilon} \right). \quad (9)$$

This can be verified as follows: first we observe that the effective resistance between two vertices of a graph $G$ is always bounded above by the diameter of $G$, see (3). Second, it was proved in [12] that w.h.p. the diameter of $G_{n,p}$ is asymptotically equal to $\frac{3\log(\varepsilon^3 n)}{\varepsilon}$ and so (9) follows immediately.

2.2 Structure of the emerging giant

Ding, Kim, Lubetzky and Peres [11] describe the following construction of a random graph, which we denote by $H$. Let $0 < \mu < 1$ satisfy $\mu e^{-\mu} = (1 + \varepsilon) e^{-(1+\varepsilon)}$. Let $\mathcal{N}(\mu, \sigma^2)$ denote the normal distribution with mean $\mu$ and variance $\sigma^2$.

GIANTCONSTRUCTION

Step 1. Let $\Lambda \sim \mathcal{N}(1 + \varepsilon - \mu, \frac{1}{\varepsilon n})$ and assign i.i.d. variables $D_u \sim \text{Poisson}(\Lambda)$ ($u \in [n]$) to the vertices, conditioned that $\sum D_u 1_{D_u \geq 3}$ is even. Let $N_k = |\{u : D_u = k\}|$ and $N_{\geq 3} = \sum_{k \geq 3} N_k$. Select a random graph $K_1$ on $N_{\geq 3}$ vertices, uniformly among all graphs with $N_k$ vertices of degree $k$ for all $k \geq 3$.

Step 2. Replace the edges of $K_1$ by paths of lengths i.i.d. $\text{Geom}(1 - \mu)$ to create $K_2$. (Hereafter, $K_1$ denotes the graph from Step 1 whose vertices are the subset of vertices of $H$ consisting of these original vertices of degree $\geq 3$ and $K_2 \supseteq K_1$ denotes the graph created by the end of this step.)

Step 3. Attach an independent $\text{Poisson}(\mu)$-Galton-Watson tree with root $v$ to each vertex $v$ of $K_2$. 

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The main result of [11] is the following theorem:

**Theorem 6.** Let \( \varepsilon \to 0 \) such that \( \varepsilon^3 n \to \infty \). For any graph property \( A \), \( \Pr(H \in A) \to 0 \) implies that \( \Pr(C_1 \in A) \to 0 \).

We will work with this construction for the remainder of the manuscript. For our application of the Gaussian free field, we make the convenient choice that \( v_0 \) is a vertex in \( K_1 \).

**Proof of Theorem 4.** Let \( H \) be the graph constructed in Steps 1-3. In view of Theorem 6, in order to show \( |E(C_1)| \approx 2\varepsilon n \), we show \( |E(H)| \approx 2\varepsilon n \). We observe that

\[
1 - \mu = \varepsilon + O(\varepsilon^2). \tag{10}
\]

Recall from Step 1 that \( \Lambda \sim N(1 + \varepsilon - \mu, \frac{1}{\varepsilon n}) \). Applying the Chebyshev inequality we see that for any \( \theta > 0 \), we have

\[
\Pr(|\Lambda - E(\Lambda)| \geq \theta) \leq \frac{1}{\theta^2 \varepsilon n}.
\]

Putting \( \theta = n^{-1/3} \), we see that \( \theta^2 \varepsilon n = \varepsilon n^{1/3} \to \infty \), so

\[
\Lambda = E\Lambda + O(n^{-1/3}) = 2\varepsilon + O(n^{-1/3} + \varepsilon^2), \quad \text{w.h.p.} \tag{11}
\]

The restriction \( \sum D_u 1_{D_u \geq 3} \) is even will be satisfied with constant probability and then we see that w.h.p.

\[
N_{\geq 3} \approx \frac{4}{3} \varepsilon^3 n \quad \text{and almost all vertices of } K_1 \text{ have degree three.} \tag{12}
\]

Therefore, w.h.p., \( |E(K_1)| \approx \frac{3}{2} \varepsilon^3 n = 2\varepsilon^3 n \).

The expected length of each path constructed by Step 2 is asymptotically equal to \( 1/(1 - \mu) \approx 1/\varepsilon \). The path lengths are independent with geometric distributions (which have exponential tails) and so their sum is concentrated around their mean (by virtue of, e.g. Bernstein’s inequality) which is asymptotically equal to \( |E(K_1)| \approx 2\varepsilon^2 n \). Thus, w.h.p., \( |E(K_2)| \approx 2\varepsilon^2 n \).

Note also that in \( K_2 \), w.h.p., there is no path longer than \( \frac{2}{\varepsilon} \log N_{\geq 3} \).

Furthermore, the expected size of each tree in Step 3 is also asymptotically equal to \( 1/\varepsilon \). These trees are independently constructed whose sizes also have exponentially decaying tails and so the total number of edges is concentrated around its mean which is asymptotically equal to \( |E(K_2)| \approx 2\varepsilon n \). Thus, w.h.p. \( |E(H)| \approx 2\varepsilon n \), which proves Theorem 4.

\[\square\]

### 2.2.1 Galton-Watson Trees

A key parameter for us will be the probability that a Galton-Watson tree with Poisson(\( \mu \)) offspring distribution survives for at least \( k \) levels. The following Lemma was proved by Ding, Kim, Lubetzky and Peres (see Lemma 4.2 in [12]).
Lemma 7. Let $0 < \mu < 1$ and $\varepsilon > 0$ satisfy $\mu e^{-\mu} = (1 + \varepsilon)e^{-(1+\varepsilon)}$. Let $T$ be a Poisson($\mu$)-Galton-Watson tree. Let $L_k$ denote the $k$-th level of $T$. Then there exist absolute constants $c_1 < c_2$ such that for any $k \geq 1/\varepsilon$ we have

$$c_1(\varepsilon \exp\{-k(\varepsilon + c_1\varepsilon^2)\}) \leq \Pr(L_k \neq \emptyset) \leq c_2(\varepsilon \exp\{-k(\varepsilon - c_2\varepsilon^2)\}).$$

Their proof also easily gives the following result.

Lemma 8. For $k < 1/\varepsilon$ we have

$$\Pr(L_k \neq \emptyset) < \frac{10}{k}.$$

We shall need the following result about trees attached in Step 3. Here and throughout the remainder of the paper,

$$N = \varepsilon^3 n.$$

Lemma 9. Consider the construction of the graph $H$ from Steps 1-3. Let $0 < \gamma < 1$. Let $\mathcal{T}$ be the set of trees attached in Step 3 of giantconstruction. Then, w.h.p. (referring to the entire construction, not just Step 3), we have

(a) there are between $\frac{1}{2} c_1 N^{1-\gamma+O(\varepsilon)}$ and $2c_2 N^{1-\gamma+O(\varepsilon)}$ trees in $\mathcal{T}$ of depth at least $\gamma \varepsilon^{-1} \log N$. \hfill (13)

(b) there are no trees in $\mathcal{T}$ of depth exceeding $\frac{2 \log N}{\varepsilon}$. \hfill (14)

Here $c_1, c_2 > 0$ are the universal constants from Lemma 7.

Proof. (a) Let $p_\gamma$ denote $\Pr(L_k \neq \emptyset)$ for $k_\gamma = \lfloor \gamma \varepsilon^{-1} \log N \rfloor$, $\gamma > 0$. Conditioning on the results of Step 1 and Step 2, the number $\nu_\gamma$ of trees created in Step 3 of depth at least $k$ is a binomial with number of trials $|V(K_2)|$ and probability of success $p_\gamma$. Recall $|V(K_2)| \approx (1 + o(1)) \frac{4N}{\varepsilon}$. It follows from Lemma 7 that

$$(1 + o(1)) \frac{4N}{3} \cdot \frac{1}{\varepsilon} \cdot c_1 \varepsilon \exp\{- (\gamma + O(\varepsilon)) \log N\} = \frac{4c_1}{3} N^{1-\gamma+O(\varepsilon)} \leq E(\nu_\gamma) \leq \frac{4c_2}{3} N^{1-\gamma+O(\varepsilon)}.$$

Since $1 - \gamma > 0$ and $\varepsilon \to 0$, note that eventually $1 - \gamma + O(\varepsilon) > \delta_0$ for some positive universal constant $\delta_0$, so $N^{1-\gamma+O(\varepsilon)} \to \infty$. 

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Thus conditional on the results of Step 1 and Step 2, \( \nu_\gamma \) is distributed as a binomial with mean going to infinity and so we have that if \( 0 < \gamma < 1 \) then the Chernoff bounds imply (13).

(b) It follows from Lemma 7 that the probability that any fixed tree has depth at least \( 2\varepsilon^{-1} \log N \) is \( O(\varepsilon N^{-2+o(1)}) \). There are w.h.p. \( O(\varepsilon^2 n) \) trees and so the expected number of trees with this or greater depth is \( O(\varepsilon^2 n \times \varepsilon N^{-2+o(1)}) = O(N^{-1+o(1)}) \). The result now follows from the Markov inequality.

\[ \square \]

### 2.3 Normal Properties

In this section we describe several properties of the normal distribution that we will use in our proof.

First suppose that \( g_1, g_2, \ldots, g_s \) are independent copies of \( N(0, 1) \). Then if \( G_s = \max_{i=1}^s g_i \),

\[
E(G_s) = \sqrt{2 \log s} - \frac{\log \log s + \log(4\pi) - 2\gamma}{\sqrt{8 \log s}} + O\left(\frac{1}{\log s}\right) \tag{15}
\]

where \( \gamma \approx 0.577 \ldots \) is the Euler-Mascheroni constant. For a proof see Cramér [10].

Next suppose that \( (X_i)_{1 \leq i \leq s} \) and \( (Y_i)_{1 \leq i \leq s} \) are two centered Gaussian vectors in \( \mathbb{R}^s \) such that \( E(X_i - X_j)^2 \leq E(Y_i - Y_j)^2 \) for all \( 1 \leq i, j \leq s \). Then,

\[
E(\max \{X_i : i = 1, 2, \ldots, s\}) \leq E(\max \{Y_i : i = 1, 2, \ldots, s\}) \tag{16}
\]

(sometimes referred to as Slepian’s lemma). See Fernique [18] (Theorem 2.1.2 and Corollary 2.1.3). Finally we have that if \( (X_i)_{1 \leq i \leq s} \) is a centered Gaussian vector and \( \sigma^2 = \max_i \text{Var}(X_i) \), then

\[
E(\max_{1 \leq i \leq s} X_i) \leq \sigma \sqrt{2 \log s}. \tag{17}
\]

This can be found, for example, in the appendix of the book by Chatterjee [3]; it follows from a simple union bound. Nevertheless, repeated carefully chosen applications of (17) will suffice to prove our upper bound on \( M \). (Importantly, observe by comparison with (15) that independent normals are asymptotically the worst case for the expected maximum.)

We also have

\[
\Pr\left(\max_{1 \leq i \leq s} X_i - E(\max_{1 \leq i \leq s} X_i) > t\right) \leq 2e^{-t^2/2\sigma^2}. \tag{18}
\]

See for example Ledoux [19].
2.4 Effective resistance on $K_2$

Recall that the emerging giant can be modeled as a collection of independent Poisson Galton Watson trees attached to $K_2$. Our proof will depend on a bound on the effective resistance of $K_2$, to show that it suffices to analyze the effective resistance within the Galton Watson trees. This is achieved by the following Lemma. Recall that we think of the graph as an electrical network where each edge is a resistor of resistance one.

**Lemma 10.** For the effective resistance on $K_2$, w.h.p., we have,

$$\max \{ R_{\text{eff}}(v, w) : v, w \in K_2 \} = O(1/\varepsilon).$$

Note that the bound in Lemma 10 is better than what one gets by simply using the diameter as in (9). It was shown in [12] that the diameter of $K_2$ is asymptotically equal to $5 \log N / 3\varepsilon$.

**Proof of Lemma 10.** We begin by shortening the induced paths between vertices created in Step 2 of Section 2.2. Let $\ell_1 = \lceil 1/\varepsilon \rceil$. We first replace a path of length $\ell$ by one of length $\lceil \ell/\ell_1 \rceil \ell_1$. Rayleigh’s Law ([15], [20]) states that increasing the resistance of any edge increases all effective resistances. Placing a vertex in the middle of an edge has the same effect as that of increasing the resistance of that edge. This implies that all resistances between vertices are increased by this change of path length. Let $R_{\text{eff}}^*$ denote the new resistances. Now every path has a length which is a multiple of $\ell_1$ and so if we replace paths, currently of length $k\ell_1$ by paths of length $k$, then we change all resistances by the same factor $\ell_1$. So, if $R_{\text{eff}}^*$ denotes these resistance then we have that

$$R_{\text{eff}}(v, w) \leq \ell_1 R_{\text{eff}}^*(v, w) \text{ for all } v, w \in K_2. \quad (19)$$

Let $K_2^* = (V^*, E^*)$ denote the graph obtained from $K_2$ in this way. Now we use the commute time identity (2) ([15], [20]) for a random walk $W^*$ on a graph $K_2^*$, to write

$$R_{\text{eff}}^*(v, w)|E^*| = \tau(v, w) + \tau(w, v), \quad (20)$$

where $\tau(v, w)$ is the expected time for $W^*$, started at $v$ to reach $w$.

Now the expected value of $\text{Geom}(p)$ is $1/p$ and so from (10) we see that the expected length of a path created in Step 2 of Section 2.2 is $\approx 1/\varepsilon$. And so the expected length of a path created for $K_2^*$ is at most 2. We then observe that if $X$ denotes the length of a path created in Step 2 then

$$\Pr(X \geq t) \leq (1 - (1 - o(1))\varepsilon)^t$$

and so w.h.p. the union bound implies that no path is of length more than $2\varepsilon^{-1} \log N$ where $N$ is as in (12). Because path lengths are independent, we see that w.h.p.

$$2N \leq |E^*| \leq (1 + o(1)) \times 2N \times 2 \leq 5N.$$
Now a simple argument based on conductance implies that w.h.p. the mixing time of $W^*$ is $\log^{O(1)} N$. Now for $v, w \in V(K_2^*)$ we see that $\tau(v, w)$ can be bounded by the mixing time plus the expected time to visit $w$ from the steady state. The latter will be at most $|E^*|/2$ and so we see from (20) that

$$\max \{ R^*_{\text{eff}}(v, w) : v, w \in K_2^* \} = O(1).$$

The lemma then follows from (19).

\[\Box\]

3 Proof of Theorem 5

Theorem 6 allows us to work with $H$ instead of $C_1$, and we assume from now on that $H$ has the following properties that have been shown or claimed to hold w.h.p. above, namely:

Assumed Properties of $H$: APOH

(i) $|V(K_1)| \approx 4N/3$,
(ii) $|E(K_1)| \approx 2N$,
(iii) $|V(K_2)| \approx 2\varepsilon^2 n$,
(iv) $|E(K_2)| \approx 2\varepsilon^2 n$,
(v) $|V(H)| \approx 2\varepsilon n$,
(vi) $|E(H)| \approx 2\varepsilon n$.
(vii) There are between $\frac{1}{2}c_1N^{1-\gamma + O(\varepsilon)}$ and $2c_2N^{1-\gamma + O(\varepsilon)}$ trees of depth at least $\gamma \varepsilon^{-1} \log N$ and there are no trees of depth exceeding $\frac{2\log N}{\varepsilon}$.

In what follows, we may write in terms of unconditional probabilities and expectations, but these will refer to the GFF and will assume that $H$ is a fixed graph with property APOH. There are some places where we have to prove further properties of $H$, but we will be sure to flag them.

3.1 Lower Bound

It turns out that for the lower bound, it suffices to consider the maximum over a very restricted set, consisting just of a single vertex from each sufficiently deep tree.
Lemma 11.

\[ \mathbb{E} \max_{v \in V(G)} \eta_v \geq (1 + o(1)) \frac{\log(\varepsilon^3 n)}{(2\varepsilon)^{1/2}}. \]

Proof. We first identify a subset of vertices on which the GFF behaves as having independent components and then produce a lower bound using Slepian’s comparison (16), combined with (15). Consider the set of Galton-Watson trees of depth at least \( d = i\varepsilon^{-1} \), \( i \) to be chosen, that are attached to a vertex within distance \( 1/\varepsilon \) of \( K_1 \) in \( H \). Choose one vertex at depth \( d \) from each tree to create \( S_d \). It follows from (13) with \( \gamma = i/\log N \), that there will be at least \( cN^{1 - \gamma + O(\varepsilon)} \) such trees for some constant \( c > 0 \). Let \( (\hat{\eta}_v)_{v \in S_d} \) be a random vector with i.i.d. \( \mathcal{N}(0, \gamma \varepsilon^{-1} \log N) \) entries. Then \( \hat{\eta}_v - \hat{\eta}_w \) has variance exactly \( 2\gamma \varepsilon^{-1} \log N \) whereas \( \eta_v - \eta_w \) has variance at least \( 2\gamma \varepsilon^{-1} \log N \) (the graph-distance between \( v \) and \( w \) is at least \( 2d = 2i\varepsilon^{-1} = 2\gamma \varepsilon^{-1} \log N \)) and so it follows from (16) that

\[ \mathbb{E}(\max \{ \hat{\eta}_v : v \in S_d \}) \geq \mathbb{E}(\max \{ \hat{\eta}_v : v \in S_d \}). \quad (21) \]

Applying (15) we see that

\[ \mathbb{E}(\max \{ \hat{\eta}_v : v \in S_d \}) \geq (1 + o(1)) (2 \log(|S_d|)^{1/2} \times (\gamma \varepsilon^{-1} \log N)^{1/2} \hat{\eta}_v \text{ has the same distribution as a standard Gaussian multiplied by } (\gamma \varepsilon^{-1} \log N)^{1/2}). \]

Using \( |S_d| \geq cN^{1 - \gamma + O(\varepsilon)} \), we obtain

\[ \mathbb{E}(\max \{ \hat{\eta}_v : v \in S_d \}) \geq (1 + o(1)) (2 \log(cN^{1 - \gamma + O(\varepsilon)}))^{1/2} \times (\gamma \varepsilon^{-1} \log N)^{1/2} \]

\[ \approx \frac{(2\gamma(1 - \gamma))^{1/2} \log N}{\varepsilon^{1/2}}. \quad (22) \]

Putting \( \gamma = 1/2 \) in (22) and applying (21) yields

\[ \mathbb{E} \max_{v \in V(G)} \eta_v \geq \mathbb{E} \max_{v \in S_d} \eta_v \geq (1 + o(1)) \frac{\log N}{(2\varepsilon)^{1/2}}. \]

Recalling that \( N = \varepsilon^3 n \), this finishes the proof of the lemma.

The important task is to achieve a matching upper bound.

3.2 Upper Bound

We begin with an outline of the proof of the upper bound.

We let \( \kappa \) denote the smallest power of 2 which is at least \( 1/\varepsilon \), and will write \( \ell_0 = \log_2 \kappa \). We let \( L_k \) denote the set of vertices at distance \( k \) from \( K_2 \). We say that \( v \in G \) is a \textit{d-survivor}
if it has at least one \( d \)-descendant \( x_d(v) \); that is, a vertex \( x_d(v) \) such that \( \text{dist}(K_2, x_d(v)) = \text{dist}(K_2, v) + \text{dist}(v, x_d(v)) = \text{dist}(K_2, v) + d \).

Finally, we set \( U^0 = K_2 \) and define for each \( 1 \leq j \leq 2 \log N \) a set \( U^j \) by choosing, for each \( \kappa \)-survivor \( v \) in \( L(j-1)\kappa \), an arbitrary \( \kappa \)-descendant \( x_\kappa(v) \). Evidently, we have for \( U = \bigcup_{j \geq 0} U^j \)

\[
\mathbb{E}(\max_{v \in \eta_u} \eta_v) \leq \mathbb{E}(\max_{u \in U} \eta_u) + \mathbb{E}(\max_{v \in V} (\eta_v - \eta_u(v))),
\]

(23)

for any function \( u : V \to U \). We will bound the two terms on the right hand side separately. Let

\[
T_\delta = \frac{\epsilon^\delta \log N}{(2\epsilon)^{1/2}},
\]

where \( \delta = o(1) \) will be chosen below in (31).

**Lemma 12.** With the notation introduced above, we have

\[
\mathbb{E}(\max_{u \in U} \eta_u) \leq (1 + o(1))T_\delta.
\]

(24)

**Lemma 13.** There is a function \( u : V \to U \) such that

\[
\mathbb{E}(\max_{v \in V} (\eta_v - \eta_u(v))) = o(T_\delta).
\]

(25)

Observe that the proof of the upper bound in Theorem 5 follows from (23) and Lemmas 12 and 13; it remains just to prove these two Lemmas.

### 3.2.1 Proof of Lemma 12

**Proof.** We let \( Z_j = \max_{v \in U^j} \eta_v \) and

\[
\mathbb{E}(\max_{v \in U} \eta_v) = \mathbb{E}\left(\max_{0 \leq j \leq 2 \log N} Z_j\right) \leq T_\delta + \sum_{j=0}^{2 \log N} \int_{t \geq T_\delta} \text{Pr}(Z_j \geq t) \, dt,
\]

(26)

and our task now is to bound the sum of integrals in (26). In words, the idea is that \( U \) is partitioned into smaller pieces \( U^j \) such that each piece is of a small enough cardinality such that the Gaussian concentration of \( Z_j \) around its mean allows to control the above integrals.

We begin with the bound on the size of each piece \( U^j \).

**Claim.**

\[
\mathbb{E}(\vert U^j \vert) \leq O\left(\epsilon^2 n \times (1 - \epsilon)^{\kappa(j-1)} \times \epsilon e^{-\epsilon^\kappa}\right) = O\left(N \epsilon e^{-\epsilon \kappa j}\right), \quad j \geq 1.
\]

(27)
Proof. We write $|U^j| = \sum_v 1_{B_v}$, where the sum is over all the vertices $v$ at level $\kappa(j-1)$ of all the G-W trees from Steps 1 and 2 and $B_v$ is the event that vertex $v$ is a $\kappa$-survivor. To bound the number of summands, we can assume that there are $O(\varepsilon^2 n)$ vertices that are roots of G-W trees i.e. are defined in Steps 1 and 2. Then the expected number of vertices at level $t$ of a G-W tree will be
\[
(1 - \varepsilon + O(\varepsilon^2))^t = O((1 - \varepsilon)^t).
\] (28)
We apply this with $t = \kappa(j - 1)$. Finally $\Pr(B_v) \leq \Pr(L_{\kappa} \neq \emptyset)$ and we use Lemma 7 to upper bound $\Pr(L_{\kappa} \neq \emptyset)$ by $O(\varepsilon e^{-\varepsilon \kappa})$.

We proceed to bound the sum in (26) term by term. (We wish to show that the sum is $o(T_{\delta}).$)

**Case 1:** $j \geq 1.$

This is a place where we have to prove additional properties of $H$. Now, if the RHS of (27) is less than $\log^2 N$ then we can use the Markov inequality to bound the size of each $|U^j|$ by $\log^4 N$ w.h.p (a union bound over $j$). On the other hand, if the RHS of (27) grows faster than $\log N$, then because we have expressed $|U^j|$ as the sum of independent Bernouilli random variables, then the Chernoff bounds mean that w.h.p. $|U^j| = O(Ne^{-\varepsilon j})$. So we bound $|U^j|$ from above by $CN e^{-\varepsilon j} + \log^4 N$. Here $C$ in (30) is a hidden constant from (27).

First we estimate the expectations.

**Claim.** For $j \geq 1,$
\[
E(Z_j) \leq e^{-\delta/2}T_{\delta}
\] (29)
with $\delta = \frac{1}{\log^{1/3} N}$.

*Proof.* For $v \in U^j$, we know that $\eta_v$ has variance $\kappa j + O(\varepsilon^{-1})$ (by the definition of $U^j$, the graph-distance from $v$ to $K_2$ is $\kappa j$ and $O(\varepsilon^{-1})$ comes from Lemma 10). It then follows from (17) in Section 2.3 and $|U^j| \leq CN e^{-\varepsilon \kappa j} + \log^4 N$ that
\[
E(Z_j) \leq (2 \log(CN e^{-\varepsilon \kappa j} + \log^4 N))^{1/2} \times (\kappa j + O(\varepsilon^{-1}))^{1/2}. 
\] (30)

We consider two cases. If $j \leq \frac{1}{100} \log N$, then (30) implies that
\[
E(Z_j) \leq (\kappa^{1/2} \log N)/9 \leq T_{\delta}/4.
\]

Otherwise, it follows from $2(xy)^{1/2} \leq x + y$ that we can write
\[
E(Z_j) \leq (2\varepsilon^{-1})^{1/2} \left( 1 + O\left( \frac{\log \log N}{\log N} \right) \right) (\kappa \varepsilon j)^{1/2} (\log N - \varepsilon \kappa j)^{1/2} 
\]
\[
\leq \left( 1 + O\left( \frac{\log \log N}{\log N} \right) \right) \sqrt{\frac{\log N}{2\varepsilon}} \leq e^{-\delta/2}T_{\delta}
\]
if we take
\[ \delta = \frac{1}{\log^{1/3} N}. \] (31)

To estimate the probability we use the Gaussian concentration for the maximum, (18) in Section 2.3, which yields
\[
\Pr(Z_j \geq \mathbf{E}(Z_j) + t) \leq 2 \exp \left\{ -\frac{t^2}{(j + O(1))\kappa} \right\} \leq 2 \exp \left\{ -\frac{t^2}{3\kappa \log N} \right\}, \quad (32)
\]
where in the last inequality we use \( j \leq 2 \log N \). Thus,
\[
\int_{t \geq T_\delta} \Pr(Z_j \geq t) dt \leq \int_{t \geq T_\delta} \exp \left\{ -\frac{(t - \mathbf{E}(Z_j))^2}{3\kappa \log N} \right\} dt
\]
\[
= \sqrt{3\kappa \log N} \int_{u \geq \frac{T_\delta - \mathbf{E}(Z_j)}{\sqrt{3\kappa \log N}}} e^{-u^2} du = O \left( \kappa^{1/2} \log^{1/2} N \exp \left\{ -\frac{(T_\delta - \mathbf{E}(Z_j))^2}{3\kappa \log N} \right\} \right). \quad (33)
\]

Plugging (29) into (33) we see that
\[
\exp \left\{ -\frac{(T_\delta - \mathbf{E}(Z_j))^2}{3\kappa \log N} \right\} \leq \exp \left\{ -\frac{(1 - e^{-\delta/2})^2 \kappa^2 T_\delta}{6 \kappa \varepsilon} \right\} \leq N^{-c \delta^2}
\]
for some universal constant \( c > 0 \), as \( \kappa \varepsilon \leq 2 \), \( e^{2\delta} \to 1 \) and \( (1 - e^{-\delta/2})^2 \approx \delta^2/4 \).

So,
\[
\int_{t \geq T_\delta} \Pr(Z_j \geq t) dt \leq \kappa^{1/2} \log^{1/2} N \times N^{-c \delta^2} \leq N^{-c \delta^2} T_\delta. \quad (34)
\]

Thus
\[
\sum_{j=1}^{2 \log N} \int_{t \geq T_\delta} \Pr(Z_j \geq t) dt \leq 2 N^{-c \delta^2} T_\delta \log N = 2 \exp \left\{ -\frac{c}{\log^{2/3} N} \log N + \log \log N \right\} T_\delta
\]
\[
= o(T_\delta). \quad (35)
\]

Case 2: \( j = 0 \).
In this case it suffices to show that \( \mathbf{E}(Z_0) = o(T_\delta) \) because it turns out that \( \mathbf{E}(Z_0) = o(T_\delta) \) implies that the first term in (26) is \( o(T_\delta) \), that is
\[
\int_{t = T_\delta}^{\infty} \Pr(Z_0 \geq t) dt = o(T_\delta).
\]
Indeed, if $\mathbf{E}(Z_0) = o(T_\delta)$, then by (18),
\[
\int_{t=T_\delta}^{\infty} \Pr(Z_0 \geq t)dt = \int_{t=T_\delta}^{\infty} \Pr(Z_0 - \mathbf{E}Z_0 \geq t - \mathbf{E}Z_0)dt
\leq \int_{t=T_\delta}^{\infty} \exp \left\{ -\frac{(t - \mathbf{E}Z_0)^2}{2(\frac{3}{\varepsilon} \log N + O(\varepsilon^{-1}))} \right\} dt,
\]
where we use that $\operatorname{Var}(\eta_v) \leq \frac{2}{\varepsilon} \log N + O(\varepsilon^{-1})$ for each $v \in U^0 = K_2$: the term $O(\varepsilon^{-1})$ comes from the resistance on $K_1$ handled in Lemma 10 and the term $\frac{2}{\varepsilon} \log N$ is justified by the fact that each edge of $K_1$ is replaced with a path no longer than $\frac{2}{\varepsilon} \log N$ to form $K_2$. If $\mathbf{E}(Z_0) = o(T_\delta)$, bounding the last integral using $\int_a^{\infty} e^{-\frac{t^2}{2\sigma^2}}dt = O(\sigma e^{-\frac{a^2}{2\sigma^2}})$ gives
\[
\int_{t=T_\delta}^{\infty} \Pr(Z_0 \geq t)dt = O \left( N^{-1/8} \sqrt{\frac{\log N}{\varepsilon}} \right) = o \left( \sqrt{\frac{\log N}{\varepsilon}} \right) = o(T_\delta).
\]
So we proceed to show that $\mathbf{E}(Z_0) = o(T_\delta)$. We have
\[
\mathbf{E}(Z_0) = \mathbf{E}(\max_{v \in K_2} \eta_v) \leq \mathbf{E}(\max_{u \in K_1} \eta_u) + \mathbf{E}(\max \min_{v \in K_2, u \in K_1} \eta_v - \eta_u).
\]

It follows from Lemma 10 that for $u_1, u_2 \in K_1$ we have $R_{\text{eff}}(u_1, u_2) \leq C/\varepsilon$ for some constant $C$. Thus by the definition of GFF (Definition 1, (iii)), (17) and our choice that $v_0 \in K_1$ we have that
\[
\mathbf{E} \left( \max_{u \in K_1} \eta_u \right) = O \left( \sqrt{2 \log(4N/3)} \sqrt{C/\varepsilon} \right) = o(T_\delta),
\]
where the size of $K_1$ is controlled by APOH (i).

To bound $\mathbf{E}(\max_{v \in K_2} \min_{u \in K_1} \eta_v - \eta_u)$ we proceed as follows. Depending on the distance of $v \in K_2$ from $K_1$, we build an efficient chain between a certain $u \in K_1$ and $v$ (to reduce the variances of the increments and bound the expected maximum using the union bound). More specifically, we consider sets $I_0, I_1, I_2, \ldots$ of pairs of vertices from $K_2$ defined by the following rule:
If $v \in K_2$ and $2^i$ is the largest power of 2 dividing $D = \text{dist}(v, K_1)$, then we add $(u, v)$ to $I_i$ where $u$ lies at distance $2^i$ from $v$ and $D - 2^i$ from $K_1$. In case there are two choices for $u$, choose one arbitrarily. Notice that $I_0$ is a subset of the edges of $K_2$.

Recall that $K_2$ has asymptotically $2\varepsilon^2 n$ vertices and edges. A path of length $\ell$ created in Step 2 of Section 2.2 contributes at most $[\ell/2^i]$ pairs to $I_i$. Thus we have w.h.p. that $|I_i| \leq 3\varepsilon^2 n/2^i$ for all $i$, say. In particular, assuming this bound (by conditioning that $C_1$ has this property) we have from (17) that
\[
\mathbf{E} \left( \max_{(v_1, v_2) \in I_i} \eta_{v_2} - \eta_{v_1} \right) \leq \sqrt{2^{i+1}} \sqrt{2 \log \left( \frac{3\varepsilon^2 n}{2^i} \right)}.
\]
Now, since each vertex \( u \in K_2 \) is joined to a vertex \( v \in K_1 \) by a path which uses at most one edge from each \( I_i \), we can bound
\[
E(\max_{u \in K_2} \min_{v \in K_1} (\eta_u - \eta_v)) \leq 2^{1/2} \sum_{i=0}^{\log(2\log N/\varepsilon)} \sqrt{2^i \log \left( \frac{3\varepsilon^2 n}{2^i} \right)}.
\] (39)
Here the upper limit of the sum comes from the fact that w.h.p. no induced path in \( K_2 \) is longer than \( 2\log N/\varepsilon \). Notice that this is essentially a chaining argument (as in Dudley’s bound, see for instance [21]).

If \( u_i \) is the summand in (39) then
\[
\frac{u_{i+1}}{u_i} = 2^{1/2} \frac{\log(3\varepsilon^2 n) - (i + 1) \log 2}{\log(3\varepsilon^2 n) - i \log 2} = 2^{1/2} \left( 1 - \frac{\log 2}{\log(3\varepsilon^2 n) - i \log 2} \right).
\]
So, if \( 2^i \leq 3\varepsilon^2 n/100 \) then \( u_{i+1}/u_i \geq 4/3 \). So, where \( 2^{i_0} \) is the largest power of 2 that is less than or equal to \( 3\varepsilon^2 n/100 \) then
\[
E(\max_{u \in K_2} \min_{v \in K_1} (\eta_u - \eta_v)) \leq 4 \times 2^{1/2} \sum_{i=i_0}^{\log(2\log N/\varepsilon)} \sqrt{2^i \log \left( \frac{3\varepsilon^2 n}{2^i} \right)} \leq O \left( \sum_{i=i_0}^{\log(2\log N/\varepsilon)} 2^i \right) = O \left( \frac{\log^{1/2} N}{\varepsilon^{1/2}} \right) = o(T_\delta). \] (40)

Plugging (38) and (40) into (37) yields \( E(Z_0) = o(T_\delta) \). It then follows from (35) and (36) and (26) that
\[
E(\max_{u \in U} \eta_u) \leq (1 + o(1)) T_\delta, \tag{41}
\]
completing the proof of (24).

3.2.2 Proof of Lemma 13

To prove Lemma 13 we let \( W_k = L_0 \cup L_k \cup L_{2k} \cup L_{3k} \cup \ldots \), for \( k = 1, 2, 4, \ldots, \kappa \) (recall \( L_0 = K_2 \)), so \( W_k \) denotes the set of vertices whose distance to \( K_2 \) is divisible by \( k \). Our goal now is to show that a general vertex \( v \) is \( \eta \)-close to some vertex \( u(v) \in U \), i.e. as measured by \( (\eta_v - \eta_u) \); we will do this by showing that \( v \) is \( \eta \)-close to its \( H \)-nearest (as measured by graph distance) ancestor \( y \in W_\kappa \); this will suffice since our choice of \( U \) ensures that some vertex \( u \in U \) has the property that \( y \) is also the \( \eta \)-closest ancestor of \( u \) in \( W_\kappa \).

We will consider sets \( J_0, J_1, J_2, \ldots, J_\ell_0 \) of ordered pairs of vertices in \( H \) with the following properties (see Figure 1):
For \((v_1, v_2) \in J_i\), we have that \(v_1, v_2 \in W_{2i}\), and that \(v_2\) is a 2\(i\)-descendant of \(v_1\).

\[W_{2i+1} \subseteq W_{2i}\]

\[W_{2i} \setminus W_{2i+1}\]

\[(v_0, x(v_0)) \in J_i\]

\[W_{2i}\]

\[v_0 \rightarrow v_1 = y(v_0)\]

\[v_0 \rightarrow x(v_0)\]

Figure 1: The sets \(W_k, J_k\).

A For \((v_1, v_2) \in J_i\), we have that \(v_1, v_2 \in W_{2i}\), and that \(v_2\) is a 2\(i\)-descendant of \(v_1\).

B \(J_0\) is the set of all edges in \(H\) that are outside of \(K_2\),

C For each \(i\), we have for each 2\(i\)-survivor \(v_0 \in W_{2i} \setminus W_{2i+1}\) that exactly one 2\(i\)-descendant \(x(v_0) \in W_{2i+1}\) of \(v_0\) is paired in \(J_{i+1}\) with its 2\(i+1\)-ancestor \(v_1 \in W_{2i+1}\).

D For all \(i\), \(\pi_2(J_{i+1}) \subset \pi_2(J_i)\). (Here \(\pi_j\) is the projection function returning the \(j\)th coordinate of a tuple.)

Notice that pairings \(J_0, J_1, \ldots, J_{i_0}\) with these properties exist by induction, and so we fix some choice of them. Indeed, given \(J_i\) we can ensure Property D by choosing \(x(v_0)\) to be a 2\(i-1\)-descendant of \(z\) where \((v_0, z) \in J_i\). We write \(\tilde{J}_i\) for the set of unordered pairs which occur (in some order) in \(J_i\). The heart of our argument is the following lemma.

**Lemma 14.** Given any vertex \(v \in V\), let \(\alpha(v)\) be its \(H\)-closest ancestor in \(W_k\). There is a sequence \(v = v_0, v_1, v_2, \ldots, v_t = \alpha(v)\) such that:

(a) For each \(j = 1, \ldots, t\), \(\{v_{j-1}, v_j\} \in \tilde{J}_i\) for some \(i\).
(b) For each \(i = 0, \ldots, \ell_0\), at most \(1 + 2(\ell_0 - i)\) of the pairs \(\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{t-1}, v_t\}\) belong to \(J_i\).

Proof of Lemma 14. Fix a vertex \(v \in V\). Our goal is to find a chain \(v = v_0, v_1, v_2, \ldots, v_t = \alpha(v)\) such that its consecutive links \(\{v_{j-1}, v_j\}\) are all in the sets \(J_i\) and each set \(J_i\) contains only at most \(1 + 2(\ell_0 - i)\) links. We shall do this recursively and in order to keep track of it, we need the following parameters

\[
\phi(v) = \max \{0 \leq i \leq \ell_0 \mid v \in W_{2^i}\} \\
\psi(v) = \max \{0 \leq i \leq \phi(v) \mid v \in \pi_2(J_i)\}.
\]

Claim 1. Given any \(v\), there is a vertex \(\alpha(v)\) such that either

(a) \(\phi(\alpha(v)) > \phi(v)\) and \((\alpha(v), v) \in J_{\phi(v)}\), or else

(b) \(\phi(\alpha(v)) = \phi(v)\) and \(\psi(\alpha(v)) > \psi(v)\), and there exists \(z(v)\) such that \((z(v), \alpha(v))\) and \((z(v), v)\) are both in \(J_{\psi(v)}\).

Proof. Consider the vertex \(v\), and let \(i = \phi(v)\). We consider two cases:

Case 1: \(\psi(v) = \phi(v)\). In this case, by definition of \(\psi(v)\), we have that there is a vertex \(\alpha(v)\) such that \((\alpha(v), v) \in J_i\). In particular, as \(2^i\) is the largest power of \(2\) such that \(v \in W_{2^i}\) and \(v\) is a \(2^i\) descendant of \(\alpha(v)\), we have that \(\alpha(v) \in W_{2^{i+1}}\); that is, that \(\phi(\alpha(v)) \geq i + 1\), as claimed.

Case 2: \(\psi(v) = j < \phi(v)\). In this case, by definition of \(\psi(v)\), we have that there is a vertex \(z\) such that \((z, v) \in J_j\). Now by Property C of the pairings \(\{J_i\}\), \(z\) has a \(2^j\)-descendant \(\alpha(v)\) which is in \(\pi_2(J_{j+1})\); in particular, we have that \(\psi(\alpha(v)) \geq j + 1 > \psi(v)\). (Note for clarity that \(\alpha(v)\) and \(v\) are at the same distance from \(K_1\) in Case 2 and so \(\phi(\alpha(v)) = \phi(v)\).) And by Property D, \(\alpha(v) \in \pi_2(J_j)\) as well, and thus \((z, \alpha(v)) \in J_j\), completing the proof of the claim. This concludes the proof of Claim 1, and thus also Lemma 14.

Observe that Lemma 14 follows from Claim 1; indeed, one can construct the claimed sequence recursively as follows: given the partially constructed sequence \(v = v_0, v_1, \ldots, v_s\) we append either the single term \(\alpha(v_s)\) or the two terms \(z(v_s), \alpha(v_s)\), according to which case of part (a) of the claim applies, and terminate if \(\phi(\alpha(v_s)) = \ell_0\). Observe that a consecutive pair \(v, v'\) in \(v_0, \ldots, v_t\) belongs (as an unordered pair) to \(J_i\) only if either

(i) \(v' = \alpha(v)\) and \(\phi(v') > \phi(v)\), or

(ii) \(v' = z(v)\), the term after \(v'\) is \(v'' = \alpha(v)\), and \(\psi(v'') > \psi(v)\), or

(iii) the term before \(v\) is \(\hat{v}\), \(v = z(\hat{v})\), \(v' = \alpha(\hat{v})\), and \(\psi(v') > \psi(\hat{v})\).
Since \((\phi(v), \psi(v))\) increases lexicographically in this way along the path, we have the claimed upper bound of \(1 + 2(\ell_0 - i)\) on the number of of consecutive pairs from \(\bar{J}_i\). This finish the proof of Lemma 14.

\(\square\)

Now we are ready to finish the proof of Lemma 13. Thanks to Lemma 14, we can decompose \(\eta_v - \eta_{\alpha(v)} = \sum_{j=1}^{\ell_0} \eta_{j-1} - \eta_j\) and using a chaining argument as before we get

\[
\mathbb{E}_{H,\eta} \left( \max_{v \in V} |\eta_v - \eta_{\alpha(v)}| \right) \leq \mathbb{E}_H \sum_{i=0}^{\ell_0} (1 + 2(\ell_0 - i)) \mathbb{E}_{\eta, \{a,b\} \in J_i} \max_a |\eta_a - \eta_b| \\
\leq O \left( \mathbb{E}_H \sum_{i=0}^{\ell_0} (\ell_0 - i + 1)\sqrt{2^i(\sqrt{2\log |J_i|})} \right).
\]

(42)

Here, \(\mathbb{E}_{H,\eta}\) is expectation over the larger space of the random graph \(H\) together with the GFF, while \(\mathbb{E}_\eta\) is the expectation of a fixed Gaussian Free Field and \(\mathbb{E}_H\) is an expectation just over the random choice of \(H\) (this is to handle \(\sqrt{\log |J_i|}\), as we do not have a h.p. statement about \(|J_i|\) covered by APOH and we will only be able to control \(\mathbb{E}_H |J_i|\)). The first inequality follows from part (b) of Lemma 14 and the second inequality follows from the union bound on the maximum, (17).

Given (42), our task is to bound \(\mathbb{E}_{H} |J_i|\) for \(0 \leq i \leq \ell_0\) and then show that the sum in (42) is \(o(T_3)\). We have from Property C and (28) that

\[
\mathbb{E}_{H} |J_i| = O \left( \mathbb{E}_H |W_{2i}| \times \frac{1}{2^i} \right) = O \left( \epsilon^2 n \times \sum_{j \geq 0} \mu^{ji} \times \frac{1}{2^i} \right) = O \left( \frac{\epsilon^2 n}{2^i(1 - \mu^i)} \right) = O \left( \frac{\epsilon n}{2^{2i}} \right)
\]

(43)

(the number of vertices on \(K_2\) is \(\epsilon^2 n\) and \(\mu^{ji}\) bounds the expected number of vertices on level \(ji\), according to (28)). Going back to (42) we see that

\[
\mathbb{E}_{H,\eta} \left( \max_{v \in V} |\eta_v - \eta_{\alpha(v)}| \right) \leq \sum_{i=0}^{\ell_0} (\ell_0 - i + 1)\sqrt{2^i \sqrt{2\log \left( \frac{\epsilon n}{2^{2i}} \right)}}.
\]

(44)

Here we use that \(\mathbb{E}_H \sqrt{\log |J_i|} \leq \sqrt{\log \mathbb{E}_H |J_i|}\), by Jensen’s inequality (\(\log^{1/2} x\) is a concave function) and (43).

It only remains to deal with the R.H.S. of (44). Given \(v \in V\), we let \(u(v)\) to be a closest vertex in \(U\) to \(v\) (in the graph distance). Suppose for now that \(u(v) = \alpha(v)\), where \(\alpha(v)\) is provided by Lemma 14.

To get a high probability result, we will use the Markov inequality: if we denote \(Y = \mathbb{E}_\eta \left( \max_{v \in V} |\eta_v - \eta_{\alpha(v)}| \right)\), we have \(\Pr_H \left( Y > (\log N)^{1/4} \mathbb{E}_H Y \right) \leq (\log N)^{-1/4}\) and this explains
the $\log^{1/4} N$ factor in (45) below. The sum in (44) can essentially be dealt with as in (39). We check that the ratio between the terms $i + 1$ and $i$ equals

$$\frac{\ell_0 - i}{\ell_0 - i + 1} \sqrt{2 \sqrt{1 - \frac{2 \log 2}{\log(\varepsilon n) - 2i \log 2}}},$$

which is strictly larger than, say $\frac{10}{9}$ for $0 \leq i \leq \ell_0 - 10$. Thus the last 10 terms dominate this sum and we get w.h.p.

$$E_{\eta \max} \max_{v \in V} |\eta_v - \eta_{\alpha(v)}| \leq O \left( \log^{1/4} N \times \sqrt{2^{\ell_0}} \sqrt{2 \log \left( \frac{\varepsilon n}{2^{2\ell_0}} \right)} \right) = O \left( \frac{\log^{3/4} N}{\varepsilon^{1/2}} \right) = o(T_\delta). \quad (45)$$

This concludes the proof of Lemma 13 in the case $u(v) = \alpha(v)$. If $u(v) \neq \alpha(v)$, then since $\eta_v - \eta_{u(v)} = (\eta_v - \eta_{\alpha(v)}) + (\eta_{\alpha(v)} - \eta_{\alpha(\alpha(v))}) + (\eta_{\alpha(\alpha(v))} - \eta_{u(v)})$, by the triangle inequality we can obtain the same bound as above up to the constant 3.

References


