On the cover time of the emerging giant

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Abstract

Let $p = \frac{1+\varepsilon}{n}$. It is known that if $N = \varepsilon^3 n \to \infty$ then w.h.p. $G_{n,p}$ has a unique giant largest component. We show that if in addition, $\varepsilon = \varepsilon(n) \to 0$ then w.h.p. the cover time of $G_{n,p}$ is asymptotic to $n \log^2 N$; previously Barlow, Ding, Nachmias and Peres had shown this up to constant multiplicative factors.

1 Introduction

Let $G = (V, E)$ be a connected graph with vertex set $V = [n] = \{1, 2, \ldots, n\}$ and an edge set E of m edges. In a simple random walk $\mathcal{W}$ on a graph G, at each step, a particle moves from its current vertex to a randomly chosen neighbor. For $v \in V$, let C_v be the expected time taken for a simple random walk starting at v to visit every vertex of G. The *vertex cover time* C_G of G is defined as $C_G = \max_{v \in V} C_v$. The (vertex) cover time of connected graphs has been extensively studied. It is a classic result of Aleliunas, Karp, Lipton, Lovász and Rackoff [1] that $C_G \leq 2m(n-1)$. It was shown by Feige [12], [13], that for any connected graph G, the cover time satisfies $(1 - o(1))n \log n \leq C_G \leq (1 + o(1))\frac{4}{27}n^3$. As an example of a graph achieving the lower bound, the complete graph K_n has cover time determined by the Coupon

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Collector problem. The *lollipop* graph consisting of a path of length $n/3$ joined to a clique of size $2n/3$ gives the asymptotic upper bound for the cover time.

Cooper and Frieze [4] established the cover time of the giant component C_1 of the random graph $G_{n,p}, p = c/n$ where $c > 1$ is a constant. They showed in this setting that w.h.p. the cover time C_{C_1} satisfies

$$C_{C_1} \approx \frac{cx(2-x)}{4(cx - \ln c)} n(\ln n)^2,$$

where x denotes the solution in $(0, 1)$ of $x = 1 - e^{-cx}$.

(Here $A_n \approx B_n$ if $A_n = (1 + o(1))B_n$ as $n \to \infty$.)

This raises the question as to what happens if $p = (1 + \varepsilon)/n$, $\varepsilon > 0$ and we allow $\varepsilon \to 0$. It is known that a unique giant component emerges w.h.p. only when $\varepsilon^3 n \to \infty$. Barlow, Ding, Nachmias and Peres [2] showed that w.h.p.

$$C_{C_1} = \Theta(n \log^2(\varepsilon^3 n)). \tag{1}$$

We prove in fact that

Theorem 1. *Suppose that $N = \varepsilon^3 n \to \infty$ and $\varepsilon \to 0$. Then w.h.p.*

$$C_{C_1} \approx n \log^2(\varepsilon^3 n).$$

This confirms a conjecture from [5], where it was shown that $C_{C_1^{(2)}} \approx \frac{\varepsilon}{4} n \log^2(\varepsilon^3 n)$ ($C_1^{(2)}$ is the 2-core of C_1, that is C_1 stripped of its attached trees). Our proof is very different from the proof in [5]. We will use the notion of a Gaussian Free Field (GFF). This was used in the breakthrough paper of Ding, Lee and Peres [9] that describes a *deterministic* algorithm for approximating C_G to within a constant factor. This was later refined by Ding [10] and by Zhai [18]. It is the latter paper that we will use. In the next section, we will describe the tools needed for our proof. Then in Section 3 we will use these tools to prove Theorem 1.

2 Tools

2.1 Gaussian Free Field

For our purposes, given a graph $G = (V, E)$, a GFF is a centered normal vector $(\eta_v, v \in V)$ where

(i) $\mathbf{E}(\eta_v) = 0$ for all $v \in V$.

(ii) $\eta_{\nu_0} = 0$ is constant for some fixed vertex $\nu_0 \in V$.

(iii) $\mathbf{E}((\eta_v - \eta_w)^2) = R_{\text{eff}}(v, w)$ for all $v, w \in V$.

Note that in particular, $\text{Var}(\eta_v) = \mathbf{E}(\eta_v^2) = R_{\text{eff}}(v, \nu_0)$. (Here R_{eff} is the effective resistance between v and w. See Doyle and Snell [11] or Lewin, Peres and Wilmer [16] for nice discussions of this notion.)

Next let

$$M = \mathbf{E}(\max_{v \in V} \eta_v).$$

Ding, Lee and Peres [9] proved that there are universal constants c_1, c_2 such that

$$c_1 |E| M^2 \leq C_G \leq c_2 |E| M^2. \tag{2}$$

Next let $R = \max_{v,w \in V} R_{\text{eff}}(v, w)$, Zhai [18] proved that there are universal constants c_3, c_4 such that if we let τ_{cov} be the first time that all the vertices in V have been visited at least once for the walk on G started at ν_0, we have

$$\mathbf{Pr}\left(\left|\tau_{cov} - |E|M^2\right| \geq |E|(\sqrt{\lambda R} \cdot M + \lambda R)\right) \leq c_3 e^{-c_4 \lambda} \tag{3}$$

for any $\lambda \geq c_3$. Setting $X = \frac{\tau_{cov}}{|E|M^2}$, this gives after crude estimates

$$|\mathbf{E}X - 1| \leq \mathbf{E}|X - 1| = \int_0^\infty \mathbf{Pr}(|X - 1| > t) dt \leq C\left(\sqrt{\frac{R}{M^2}} + \frac{R}{M^2}\right)$$

for a universal constant C. Since R and M do not depend on ν_0, after taking the maximum over ν_0 we thus get that $C_G = \max_{\nu_0} \mathbf{E}\tau_{cov}$ satisfies

$$C_G = |E|M^2\left(1 + O\left(\sqrt{\frac{R}{M^2}} + \frac{R}{M^2}\right)\right).$$

Now, as we will see in the next section, the number of edges in the emerging giant satisfies

$$|E| \approx 2\varepsilon n \quad w.h.p. \tag{4}$$

We can therefore prove Theorem 1 by showing that in the case of the emerging giant we have w.h.p. that

$$R = o(M^2) \text{ and } M \approx \frac{\log(\varepsilon^3 n)}{(2\varepsilon)^{1/2}}. \tag{5}$$

Now we know from (1), (2) and (4) that w.h.p. $M = \Omega(\varepsilon^{-1/2} \log(\varepsilon^3 n))$. Therefore to prove that $R = o(M^2)$ it will be sufficient to prove

$$R = O\left(\frac{\log(\varepsilon^3 n)}{\varepsilon}\right). \tag{6}$$

2.2 Structure of the emerging giant

Ding, Kim, Lubetzky and Peres [7] describe the following construction of a random graph, which we denote by H. Let $0 < \mu < 1$ satisfy $\mu e^{-\mu} = (1+\varepsilon)e^{-(1+\varepsilon)}$. Let $\mathcal{N}(\mu, \sigma^2)$ denote the normal distribution with mean μ and variance σ^2.

Step 1. Let $\Lambda \sim \mathcal{N}\left(1+\varepsilon-\mu, \frac{1}{\varepsilon n}\right)$ and assign i.i.d. variables $D_u \sim Poisson(\Lambda)$ $(u \in [n])$ to the vertices, conditioned that $\sum D_u 1_{D_u \geq 3}$ is even.
Let $N_k = |\{u : D_u = k\}|$ and $N_{\geq 3} = \sum_{k \geq 3} N_k$. Select a random graph K_1 on $N_{\geq 3}$ vertices, uniformly among all graphs with N_k vertices of degree k for $k \geq 3$.

Step 2. Replace the edges of K_1 by paths of lengths i.i.d. $Geom(1-\mu)$ to create K_2. (Hereafter, K_1 denotes the subset of vertices of H consisting of these original vertices of degree ≥ 3 and $K_2 \supseteq K_1$ denotes the vertices created by the end of this step.)

Step 3. Attach an independent Poisson(μ)-Galton-Watson tree to each vertex of K_2.

The main result of [7] is that for any graph property $\mathcal{A}$, $\mathbf{Pr}(H \in \mathcal{A}) \to 0$ implies that $\mathbf{Pr}(C_1 \in \mathcal{A}) \to 0$, so we work with this construction for the remainder of the manuscript. For our application of the Gaussian free field, we make the convenient choice that ν_0 is a vertex in K_1.

We next observe that

$$1 - \mu = \varepsilon + O(\varepsilon^2). \tag{7}$$

Applying the Chebyshev inequality we see that for any $\theta > 0$ we have

$$\mathbf{Pr}\left(|\Lambda - \mathbf{E}(\Lambda)| \geq \theta\right) \leq \frac{1}{\theta^2 \varepsilon n}.$$

Putting $\theta = N^{-1/3}\varepsilon$ (re-call that $N = \varepsilon^3 n$) we see that

$$\Lambda = 2\varepsilon + O(\varepsilon N^{-1/3} + \varepsilon^2), \qquad w.h.p. \tag{8}$$

The restriction $\sum D_u 1_{D_u \geq 3}$ is even will be satisfied with constant probability and then we see that w.h.p.

$$N_{\geq 3} \approx \frac{4\varepsilon^3 n}{3} = \frac{4N}{3} \text{ and almost all vertices of } K_1 \text{ have degree three.} \tag{9}$$

The expected length of each path constructed by Step 2 will be asymptotically equal to $1/(1-\mu) \approx 1/\varepsilon$. The path lengths are independent and so their sum will be concentrated around their mean which is asymptotically equal to $2\varepsilon^2 n$. Finally w.h.p. there will be no path longer than $2 \log N/\varepsilon$.

Furthermore, the expected size of each tree in Step 3 is also asymptotically equal to $1/\varepsilon$. These trees are independently constructed and so the total number of edges is concentrated around its mean which is asymptotically equal to $2\varepsilon n$. This justifies (4).

2.3 Normal Properties

In this section we describe several properties of the normal distribution that we will use in our proof.

First suppose that $g_1, g_2, \ldots, g_s$ are independent copies of $\mathcal{N}(0,1)$. Then if $G_s = \max_{i=1,\ldots,s} g_i$,

$$\mathbf{E}\left(G_s\right) = \sqrt{2\log s} - \frac{\log\log s + \log(4\pi) - 2\gamma}{\sqrt{8\log s}} + O\left(\frac{1}{\log s}\right) \tag{10}$$

where $\gamma = 0.577\ldots$ is the Euler-Mascheroni constant. For a proof see Cramér [6].

Next suppose that (X_i) and (Y_i) $1 \le i \le s$ are two centered Gaussian vectors in R^n such that $\mathbf{E}(X_i - X_j)^2 \le \mathbf{E}(Y_i - Y_j)^2$ for all $1 \le i, j \le s$. Then,

$$\mathbf{E}(\max\{X_i : i = 1, 2, \ldots, s\}) \le \mathbf{E}(\max\{Y_i : i = 1, 2, \ldots, s\}). \tag{11}$$

See Fernique [14], (Theorem 2.1.2 and Corollary 2.1.3). Finally we have that if $(X_i)_{1\le i\le s}$ is a centered Gaussian vector and $\sigma^2 = \max_i \mathbf{Var}(X_i)$, then

$$\mathbf{E}(\max_{1\le i\le s} X_i) \le \sigma\sqrt{2\log s}. \tag{12}$$

This can be found, for example, in the appendix of the book by Chatterjee [3]; it follows from a simple union bound. Nevertheless, repeated carefully chosen applications of (12) will suffice to prove our upper bound on M. (Importantly, recall by comparison with (10) that independent normals are the asymptotically the worst case for the expected max.)

We also have

$$\mathbf{Pr}(|\max_{1\le i\le s} X_i - \mathbf{E}(\max_{1\le i\le s} X_i)| > t) \le 2e^{-t^2/2\sigma^2}. \tag{13}$$

See for example Ledoux [15].

2.4 Galton-Watson Trees

A key parameter for us will be the probability that a Galton-Watson tree with Poisson(μ) offspring distribution survives for at least k levels. The following Lemma was proved by Ding, Kim, Lubetzky and Peres (see Lemma 4.2 in [8]).

Lemma 2. *Let μ be as in Section 2.2 and let T be a Galton-Watson tree added in Step 3. Let L_k denote the k-th level of T. For any $k \ge 1/\varepsilon$ we have*

$$\mathbf{Pr}\left(L_k \ne \emptyset\right) = \Theta(\varepsilon \exp\left\{-k(\varepsilon + O(\varepsilon^2))\right\}).$$

Their proof also easily gives:

Lemma 3. *For $k < 1/\varepsilon$ we have*

$$\mathbf{Pr}\left(L_k \neq \emptyset\right) < \frac{10}{k}.$$

It follows from Lemma 2 that the expected number of trees created in Step 3 of depth at least $\gamma\varepsilon^{-1}\log N$, $\gamma \geq 1/\log N$ lies between $c_1 N \times \varepsilon^{-1} \times \varepsilon \exp\{-(\gamma \log N + O(\varepsilon \log N))\} = c_1 N^{1-\gamma+O(\varepsilon)}$ and $c_2 N^{1-\gamma+O(\varepsilon)}$ for some constants $0 < c_1 < c_2$.

Conditioning on the results of Step 1 and Step 2, the number of such trees is distributed as a binomial with mean going to infinity and so we have that if $0 < \gamma < 1$ then we have the following:

$$\text{W.h.p. there are between } \frac{1}{2}c_1 N^{1-\gamma+O(\varepsilon)} \text{ and } 2c_2 N^{1-\gamma+O(\varepsilon)} \text{ trees of depth at least } \gamma\varepsilon^{-1}\log N. \tag{14}$$

The probability that any fixed tree has depth at least $2\varepsilon^{-1}\log N$ is $O(\varepsilon N^{-2+o(1)})$. There are w.h.p. $O(\varepsilon^2 n)$ trees and so the expected number of trees with this or greater depth is $O(\varepsilon^2 n \times \varepsilon N^{-(2+o(1))}) = O(N^{-(1+o(1))})$. We therefore have the following.

$$\text{W.h.p. there are no trees of depth exceeding } \frac{2\log N}{\varepsilon}. \tag{15}$$

3 Proof of Theorem 1

3.1 Effective resistance on the kernel

We begin by estimating the effective resistance between vertices of the *kernel* K_1. This is needed to justify (6).

We begin by shortening the induced paths between vertices created in Step 2 of Section 2.2. Let $\ell_1 = \lceil 1/\varepsilon \rceil$. We first replace a path of length ℓ by one of length $\lceil \ell/\ell_1 \rceil \ell_1$. Rayleigh's Law ([11], [16]) implies that this increases all resistances between vertices. Let $\widehat{R}_{\text{eff}}$ denote the new resistances. Now every path has a length which is a multiple of ℓ_1 and so if we replace paths, currently of length $k\ell_1$ by paths of length k, then we change all resistances by the same factor ℓ_1. So, if R^*_{eff} denotes these resistance then we have that

$$R_{\text{eff}}(v, w) \leq \ell_1 R^*_{\text{eff}}(v, w) \text{ for all } v, w \in K_1. \tag{16}$$

Let $K_1^* = (V^*, E^*)$ denote the graph obtained from K_1 in this way. Now we use the commute time identity ([11], [16]) for a random walk $\mathcal{W}^*$ on a graph K_1^*.

$$R^*_{\text{eff}}(v, w)|E^*| = \tau(v, w) + \tau(w, v), \tag{17}$$

where $\tau(v,w)$ is the expected time for $\mathcal{W}^*$, started at v to reach w.

Now the expected length of a path created in Step 2 of Section 2.2 is $\approx 1/\varepsilon$ and so the expected length of a path created for K_1^* is at most 2. We then observe that if X denotes the length of a path created in Step 2 then

$$\mathbf{Pr}(X \geq t) \leq (1-(1-o(1))\varepsilon)^t$$

and so w.h.p. the union bound implies that no path is of length more than $2\varepsilon^{-1}\log N$ where N is as in (9). Because path lengths are independent, we see that w.h.p.

$$2N \leq |E^*| \leq (1+o(1)) \times 2N \times 2 \leq 5N.$$

Now a simple argument based on conductance implies that w.h.p. the mixing time of $\mathcal{W}^*$ is $\log^{O(1)} N$. Now for $v,w \in V(K_1^*)$ we see that $\tau(v,w)$ can be bounded by the mixing time plus the expected time to visit w from the steady state. The latter will be at most $|E^*|/2$ and so we see from (17) that

$$\max\{R^*_{\text{eff}}(v,w) : v,w \in K_1\} = O(1).$$

It then follows from (16) that

$$\max\{R_{\text{eff}}(v,w) : v,w \in K_1\} = O(1/\varepsilon). \tag{18}$$

Together with (15), this verifies (6).

From now on, we condition on C_1 having the required properties and work in the probability space defined by the GFF, with the one exception in equation (37).

3.2 Lower Bound

To prove Theorem 1 the main task is to determine the expected maximum η_v. It turns out that for the lower bound, it suffices to consider the maximum over a very restricted set, consisting just of a single vertex from each sufficiently deep tree.

Consider the set of Galton-Watson trees of depth at least $d = i\varepsilon^{-1}$, i to be chosen, that are attached to a vertex within distance $1/\varepsilon$ of K_1 in G. Choose one vertex at depth d from each tree to create S_d. It follows from (14) with $\gamma = i/\log N$, that there will be $\approx cN^{1-\gamma+O(\varepsilon)}$ such trees for some constant $c > 0$. Let $(\widehat{\eta}_v)_{v\in S_d}$ be a random vector with i.i.d. $\mathcal{N}(0, \gamma\varepsilon^{-1}\log N)$ components. Then $\widehat{\eta}_v - \widehat{\eta}_w$ has variance exactly $2\gamma\varepsilon^{-1}\log N$ whereas $\eta_v - \eta_w$ has variance at least $2\gamma\varepsilon^{-1}\log N$ and so it follows from (11) that

$$\mathbf{E}(\max\{\eta_v : v \in S_d\}) \geq \mathbf{E}(\max\{\widehat{\eta}_v : v \in S_d\}). \tag{19}$$

Applying (10) we see that

$$\mathbf{E}(\max\{\widehat{\eta}_v : v \in S_d\}) \geq (1+o(1))(2\log(cN^{1-\gamma+O(\varepsilon)})^{1/2} \times (\gamma\varepsilon^{-1}\log N)^{1/2} \approx \frac{(2\gamma(1-\gamma))^{1/2}\log N}{\varepsilon^{1/2}}. \quad (20)$$

Putting $\gamma = 1/2$ in (20) and applying (19) yields a lower bound for $M = \mathbf{E}(\max\{\eta_v : v \in V\})$ sufficient for (5). It remains to determine a matching upper bound.

3.3 Upper Bound

We let κ denote the smallest power of 2 which is at least $1/\varepsilon$, and will write $\ell_0 = \log_2 \kappa$. We let L_k denote the set of vertices at distance k from K_2. We say that $v \in G$ is a *d-survivor* if it has at least one d-descendant $x_d(v)$; that is, a vertex $x_d(v)$ such that $\text{dist}(K_2, x_d(v)) = \text{dist}(K_2, v) + \text{dist}(v, x_d(v)) = \text{dist}(K_2, v) + d$.

Finally, we set $U^0 = K_2$ and define for each $1 \leq j \leq 2\log N$ a set U^j by choosing, for each κ-survivor v in $L_{(j-1)\kappa}$, an arbitrary κ-descendant $x_\kappa(v)$. Evidently, we have for $U = \bigcup_{j\geq 0} U^j$ that

$$\mathbf{E}(\max_{v\in V}\eta_v) \leq \mathbf{E}(\max_{u\in U}\eta_u) + \mathbf{E}(\max_{v\in V}(\eta_v - \eta_{u(v)})), \quad (21)$$

for any function $u : V \to U$. We will bound the two terms on the righthand side separately.

We begin with the first term. Let

$$T_\delta = \frac{e^\delta \log N}{(2\varepsilon)^{1/2}}$$

where $\delta = o(1)$ will be chosen below in (28). We then let $Z_j = \max_{v\in U^j}\eta_v$ and

$$\mathbf{E}(\max_{v\in U}\eta_v) = \mathbf{E}\left(\max_{0\leq j\leq 2\log N} Z_j\right) \leq T_\delta + \sum_{j=0}^{2\log N}\int_{t\geq T_\delta}\mathbf{Pr}(Z_j \geq t)dt. \quad (22)$$

Now we have, where we write $A \leq_O B$ in place of $A = O(B)$,

$$\mathbf{E}(|U^j|) \leq_O \varepsilon^2 n \times (1-\varepsilon)^{\kappa(j-1)} \times \varepsilon e^{-\varepsilon\kappa} \leq Ne^{-\varepsilon\kappa j}, \qquad j \geq 1. \quad (23)$$

Explanation: We can assume that there are $O(\varepsilon^2 n)$ vertices that are roots of G-W trees i.e. are defined in Steps 1 and 2. Then the expected number of vertices at level $\kappa(j-1)$ of a G-W tree will be $(1-\varepsilon+O(\varepsilon^2))^{\kappa(j-1)} = O((1-\varepsilon)^{\kappa(j-1)})$. Then we use Lemma 2 to bound the number of κ-survivors.

Case 1: $j \geq 1$.
Now, assuming that the RHS of (23) grows faster than $\log N$, we can assume that $|U^j| \leq_O$

$Ne^{-\varepsilon\kappa j}$. Furthermore, if this expression is less than $\log^2 N$ then we can use the Markov inequality to bound the size of $|U^j|$ by $\log^4 N$.

Now, if $v \in U^j$ then η_v has variance $\kappa j + O(\varepsilon^{-1})$. It then follows from Section 2.3 that

$$\mathbf{E}(Z_j) \leq (2\log(CNe^{-\varepsilon\kappa j} + \log^4 N))^{1/2} \times (\kappa j + O(\varepsilon^{-1}))^{1/2}. \tag{24}$$

$$\mathbf{Pr}(Z_j \geq \mathbf{E}(Z_j) + t) \leq 2\exp\left\{-\frac{t^2}{(j+O(1))\kappa}\right\} \leq 2\exp\left\{-\frac{t^2}{3\kappa\log N}\right\}. \tag{25}$$

Here C in (24) is a hidden constant from (23).

$$\int_{t\geq T_\delta} \mathbf{Pr}(Z_j \geq t)dt \leq \int_{t\geq T_\delta} \exp\left\{-\frac{(t-\mathbf{E}(Z_j))^2}{3\kappa\log N}\right\} dt \leq_O \kappa^{1/2}\log^{1/2} N \exp\left\{-\frac{(T_\delta-\mathbf{E}(Z_j))^2}{3\kappa\log N}\right\}. \tag{26}$$

Now if $j \leq \frac{1}{100}\log N$ then (24) implies that $\mathbf{E}(Z_j) \leq (\kappa^{1/2}\log N)/9 \leq T_\delta/4$ and similarly for $\frac{99}{100}\log N \leq j \leq 2\log N$. Otherwise, it follows from $2(xy)^{1/2} \leq x + y$ that we can write

$$\mathbf{E}(Z_j) \leq (2\varepsilon^{-1})^{1/2}\left(1 + O\left(\frac{\log\log N}{\log N}\right)\right)(\kappa\varepsilon j)^{1/2}(\log N - \varepsilon\kappa j)^{1/2} \leq \left(1 + O\left(\frac{\log\log N}{\log N}\right)\right)\frac{\log N}{(2\varepsilon)^{1/2}} \leq e^{-\delta/2}T_\delta, \tag{27}$$

if we take

$$\delta = \frac{1}{\log^{1/3} N}. \tag{28}$$

Plugging this into (26) we see that

$$\int_{t\geq T_\delta} \mathbf{Pr}(Z_j \geq t)dt \leq \kappa^{1/2}\log^{1/2} N \times N^{-\Omega(\delta^2)} \leq N^{-\Omega(\delta^2)}T_\delta. \tag{29}$$

Thus

$$\sum_{j=1}^{2\log N}\int_{t\geq T_\delta} \mathbf{Pr}(Z_j \geq t)dt \leq o(T_\delta). \tag{30}$$

Case 2: $j = 0$**.**
It suffices to show that $\mathbf{E}(Z_0) = o(T_\delta)$ because then by (13),

$$\int_{t=T_\delta}^{\infty} \mathbf{Pr}(Z_0 \geq t)dt \leq \int_{t=T_\delta}^{\infty} \exp\left\{-\frac{(t-\mathbf{E}Z_0)^2}{2(\frac{2}{\varepsilon}\log N + O(\varepsilon^{-1}))}\right\} dt = o\left(\sqrt{\frac{\log N}{\varepsilon}}\right) \tag{31}$$

(by (18) and the fact that there are no paths longer than $\frac{2}{\varepsilon}\log N$, for every $v \in U^0$, η_v has variance $\frac{2}{\varepsilon}\log N + O(\varepsilon^{-1})$).

We have

$$\mathbf{E}(Z_0) \leq \mathbf{E}(\max_{v\in K_1} \eta_v) + \mathbf{E}(\max_{u\in K_2}\min_{v\in K_1} \eta_u - \eta_v).$$

It follows from (18) that for $v_1, v_2 \in K_1$ we have $R_{\text{eff}}(u, v) \leq C/\varepsilon$ for some constant C. Thus by (12) and our choice that $\nu_0 \in K_1$ we have that

$$\mathbf{E}\left(\max_{v\in K_1} \eta_v\right) \leq_O \sqrt{2\log(2N)}\sqrt{C/\varepsilon}. \tag{32}$$

To bound $\mathbf{E}(\max_{v\in K_2}\min_{u\in K_1} \eta_u - \eta_v)$ we proceed as follows. We consider sets $I_0, I_1, I_2, \ldots$ of pairs of vertices from K_2 defined by the following rule:
For $v \in K_2$, if 2^i is the largest power of 2 dividing $D = \text{dist}(v, K_1)$, then we add (u, v) to I_i for a single vertex u lying at distance 2^i from v and $D - 2^i$ from K_1. Notice that I_0 is simply the set of all edges of K_2.

Recall that K_2 has asymptotically $2\varepsilon^2 n$ vertices; thus we have w.h.p. that $|I_i| \leq 3\varepsilon^2 n/2^i$ for all i, say. In particular, assuming this bound (by conditioning that C_1 has this property) we have that

$$\mathbf{E}\left(\max_{(v_1,v_2)\in I_i} \eta_{v_2} - \eta_{v_1}\right) \leq \sqrt{2^i}\sqrt{2\log\left(\frac{3\varepsilon^2 n}{2^i}\right)}.$$

Now, since each vertex $u \in K_2$ is joined to a vertex $v \in K_1$ by a path which uses at most one edge from each I_i, we can bound

$$\mathbf{E}(\max_{u\in K_2}\min_{v\in K_1} \eta_u - \eta_v) \leq_O \sum_{i=0}^{\log(2\log N/\varepsilon)} \sqrt{2^i \log\left(\frac{3\varepsilon^2 n}{2^i}\right)}. \tag{33}$$

Here the upper limit of the sum comes from the fact that w.h.p. no induced path in K_2 is longer than $2\log N/\varepsilon$. Notice that this is essentially a simple chaining argument (as in Dudley's bound, see for instance [17]).

If u_i is the summand in (33) then

$$\frac{u_{i+1}}{u_i} = 2^{1/2}\frac{\log(3\varepsilon^2 n) - (i+1)\log 2}{\log(3\varepsilon^2 n) - i\log 2} = 2^{1/2}\left(1 - \frac{\log 2}{\log(3\varepsilon^2 n) - i\log 2}\right).$$

So, if $2^i \leq 3\varepsilon^2 n/100$ then $u_{i+1}/u_i \geq 4/3$. So, where 2^{i_0} is the largest power of 2 that is less than or equal to $3\varepsilon^2 n/100$ then

$$\mathbf{E}(\max_{u\in K_2}\min_{v\in K_1}\eta_u-\eta_v)\leq_O \sum_{i=i_0}^{\log(2\log N/\varepsilon)}\sqrt{2^i\log\left(\frac{3\varepsilon^2 n}{2^i}\right)}\leq_O \sum_{i=i_0}^{\log(2\log N/\varepsilon)} 2^{i/2}\leq_O \frac{\log^{1/2}N}{\varepsilon^{1/2}}=o(T_\delta). \quad (34)$$

Combining (32) and (34) yields $\mathbf{E}(Z_0)=o(T_\delta)$. Now it follows from (30) and (31) that

$$\mathbf{E}(\max_{u\in U}\eta_u)\leq(1+o(1))T_\delta. \quad (35)$$

Now let us bound the second term on the righthand side of (21). For this purpose we let $W_k=L_k\cup L_{2k}\cup L_{3k}\cup\dots$ denote the set of vertices whose distance to K_2 is divisible by k. Our goal now is to show that a general vertex v is close to some vertex $u(v)\in U$ as measured by $(\eta_v-\eta_u)$; we will do this by showing that v is close to its nearest (in graph distance) ancestor $y\in W_\kappa$; this will suffice since our choice of U ensures that some vertex $u\in U$ has the property that y is also the closest ancestor of u in W_κ.

We will consider sets $J_0,J_1,J_2,\dots,J_{\ell_0}$ of ordered pairs of vertices in G with the following properties:

1. For $(v_1,v_2)\in J_i$, we have that $v_1,v_2\in W_{2^i}$, and that v_2 is a 2^i-descendant of v_1.
2. J_0 is the set of all edges in G that are outside of K_2,
3. For each i, we have for each 2^i-survivor $v_0\in W_{2^i}\setminus W_{2^{i+1}}$ that exactly one 2^i-descendant $x(v_0)\in W_{2^{i+1}}$ of v_0 is paired in J_{i+1} with it's 2^{i+1}-ancestor $v_1\in W_{2^{i+1}}$.
4. For all i, $\pi_2(J_{i+1})\subset\pi_2(J_i)$. (Here π_j is the projection function returning the jth coordinate of a tuple.)

Notice that pairings $J_0,J_1,\dots,J_{\ell_0}$ with these properties exist by induction, and so we fix some choice of them. We write $\bar{J}_i$ for the set of unordered pairs which occur (in some order) in J_i. The following simple observation is essential to our argument:

Lemma 4. *Given any vertex in $v\in V$, whose closest ancestor in W_κ is $\alpha(v)$, we have that there is a sequence $v=v_0,v_1,v_2,\dots,v_t=\alpha(v)$ such that:*

(a) For each $j=1,\dots,t$, $\{v_{j-1},v_j\}\in\bar{J}_i$ for some i.

(b) For each $i=0,\dots,\ell_0$, at most $1+2(\ell_0-i)$ of the pairs $\{v_0,v_1\},\{v_1,v_2\},\dots,\{v_{t-1},v_t\}$ belong to $\bar{J}_i$.

Proof. Given a vertex v, we define the parameters

$$\phi(v) = \max\{0 \le i \le \ell_0 \mid v \in W_{2^i}\}$$
$$\psi(v) = \max\{0 \le i \le \phi(v) \mid v \in \pi_2(J_i)\}.$$

We claim that given any v, there is a vertex $a(v)$ such that either

(a) $\phi(a(v)) > \phi(v)$ and $(a(v), v) \in J_{\phi(v)}$, or else

(b) $\phi(a(v)) = \phi(v)$ and $\psi(a(v)) > \psi(v)$, and there exists $z(v)$ such that $(z(v), a(v))$ and $(z(v), v)$ are both in $J_{\psi(v)}$ for some i,

Observe that the Lemma follows from the claim; indeed, one can construct the claimed sequence recursively as follows: given the partially constructed sequence $v = v_0, v_1, \ldots, v_s$ we append either the single term $a(v_s)$ or the two terms $z(v_s), a(v_s)$, according to which case of part (a) of the claim applies, and terminate if $\phi(a(v_s)) = \ell_0$. Observe that a consecutive pair v, v' in $v_0, \ldots, v_t$ only belongs (as an unordered pair) to $\bar{J}_i$ only if either

(i) $v' = a(v)$ and $\phi(v') > \phi(v)$, or

(ii) $v' = z(v)$, the term after v' is $v'' = a(v)$, and $\psi(v'') > \psi(v)$, or

(iii) the term before v is $\hat{v}$, $v = z(\hat{v})$, $v' = a(\hat{v})$, and $\psi(v') > \psi(\hat{v})$.

Since $(\phi(v), \psi(v))$ increases lexicographically in this way along the path, we have the claimed upper bound of $1 + 2(\ell_0 - i)$ on the number of of consecutive pairs from $\bar{J}_i$.

To prove the claim, consider the vertex v, and let $i = \phi(v)$. We consider two cases:
Case 1: $\psi(v) = \phi(v)$. In this case, by definition of $\psi(v)$, we have that there is a vertex $a(v)$ such that $(a(v), v)$ in J_i. In particular, as 2^i is the largest power of 2 in such that $v \in W_{2^i}$ and v is a 2^i descendant of $a(v)$, we have that $a(v) \in W_{2^{i+1}}$; that is, that $\phi(a(v)) \ge i + 1$, as claimed.
Case 2: $\psi(v) = j < \phi(v)$. In this case, by definition of $\psi(v)$, we have that there is a vertex z such that (z, v) in J_j. Now by Property 3 of the pairings $\{J_i\}$, z has a 2^j-descendant $a(v)$ which is in $\pi_2(J_{j+1})$; in particular, we have that $\psi(a(v)) \ge j + 1 > \psi(v)$. (Note for clarity that $a(v)$ and v are at the same distance from K_1 in Case 2 and so $\phi(a(v)) = \phi(v)$.) And by Property 4, $a(v) \in \pi_2(J_i)$ as well, and thus $(z, a(v)) \in J_i$, completing the proof of the claim. $\square$

Our next task is to bound $|J_i|$ for $0 \le i \le \ell_0$. We have from Property 3 and Lemma 3 that

$$\mathbf{E}|J_i| \le_O \mathbf{E}|W_{2^i}| \times \frac{1}{2^i} \le_O \sum_{j \ge 0} \frac{\varepsilon^2 n \mu^{ji}}{2^i} \le_O \frac{\varepsilon^2 n}{2^i(1 - \mu^i)} \le_O \frac{\varepsilon n}{2^{2i}}. \quad (36)$$

It remains to show that the second term in (21) is $o(T_\delta)$. Recall that given $v \in V$, we choose $u(v)$ to be a close vertex in U to v (in the graph distance). Without loss of generality we can assume that $u(v) = \alpha(v)$, where $\alpha(v)$ is provided by Lemma 4, because otherwise, since $\alpha(u(v)) = \alpha(\alpha(v))$, we write $\eta_v - \eta_{u(v)} = (\eta_v - \eta_{\alpha(v)}) + (\eta_{\alpha(v)} - \eta_{\alpha(\alpha(v))}) + (\eta_{\alpha(u(v))} - \eta_{u(v)})$ and by the triangle inequality we can obtain the same bound as below up to the constant 3. Thanks to Lemma 4, we decompose $\eta_v - \eta_{\alpha(v)} = \sum_{j=1}^{t} \eta_{j-1} - \eta_j$ and using a chaining argument as before we get

$$\begin{aligned} \mathbf{E}_{H,\eta}\left(\max_{v\in V}|\eta_v - \eta_{\alpha(v)}|\right) &\le \mathbf{E}_H \sum_{i=0}^{\ell_0}(1+2(\ell_0 - i))\mathbf{E}_\eta \max_{\{a,b\}\in \bar{J}_i} |\eta_a - \eta_b| \\ &\le_O \mathbf{E}_H \sum_{i=0}^{\ell_0}(\ell_0 - i + 1)\sqrt{2^i}(\sqrt{2\log|J_i|}), \qquad (37) \\ &\le_O \sum_{i=0}^{\ell_0}(\ell_0 - i + 1)\sqrt{2^i}\sqrt{2\log\left(\frac{\varepsilon n}{2^{2i}}\right)}. \end{aligned}$$

Here, $\mathbf{E}_{H,\eta}$ is expectation over the larger space of the random graph H together with the GFF, while $\mathbf{E}_\eta$ is the expectation of a fixed Gaussian Free Field and $\mathbf{E}_H$ is an expectation just over the random choice of H. In the last inequality we use (12) and Jensen's inequality and the fact that $\log^{1/2} x$ is a concave function. To get a high probability result, we will use the Markov inequality and this explains the $\log^{1/4} N$ factor in (38) below. The last sum can essentially be dealt with as in (33). We check that the ratio between the terms $i+1$ and i equals

$$\frac{\ell_0 - i}{\ell_0 - i + 1}\sqrt{2}\sqrt{1 - \frac{2\log 2}{\log(\varepsilon n) - 2i\log 2}}$$

which is strictly larger than, say $\frac{10}{9}$ for $0 \le i \le \ell_0 - 10$. Thus the last 10 terms dominate this sum and we get w.h.p.

$$\mathbf{E}_\eta \max_{v\in V}|\eta_v - \eta_{\alpha(v)}| \le \log^{1/4} N \times \sqrt{2^{\ell_0}}\sqrt{2\log\left(\frac{\varepsilon n}{2^{2\ell_0}}\right)} \le_O \frac{\log^{3/4} N}{\varepsilon^{1/2}} = o(T_\delta). \qquad (38)$$

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