On the cover time of the emerging giant

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Abstract

Let \( p = \frac{1+\varepsilon}{n} \). It is known that if \( N = \varepsilon^3 n \to \infty \) then w.h.p. \( G_{n,p} \) has a unique giant largest component. We show that if in addition, \( \varepsilon = \varepsilon(n) \to 0 \) then w.h.p. the cover time of \( G_{n,p} \) is asymptotic to \( n \log^2 N \); previously Barlow, Ding, Nachmias and Peres had shown this up to constant multiplicative factors.

1 Introduction

Let \( G = (V,E) \) be a connected graph with vertex set \( V = [n] = \{1,2,\ldots,n\} \) and an edge set \( E \) of \( m \) edges. In a simple random walk \( \mathcal{W} \) on a graph \( G \), at each step, a particle moves from its current vertex to a randomly chosen neighbor. For \( v \in V \), let \( C_v \) be the expected time taken for a simple random walk starting at \( v \) to visit every vertex of \( G \). The vertex cover time \( C_G \) of \( G \) is defined as \( C_G = \max_{v \in V} C_v \). The (vertex) cover time of connected graphs has been extensively studied. It is a classic result of Aleliunas, Karp, Lipton, Lovász and Rackoff [1] that \( C_G \leq 2m(n-1) \). It was shown by Feige [12], [13], that for any connected graph \( G \), the cover time satisfies \( (1-o(1))n \log n \leq C_G \leq (1+o(1))\frac{4}{27}n^3 \). As an example of a graph achieving the lower bound, the complete graph \( K_n \) has cover time determined by the Coupon

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Collector problem. The *lollipop* graph consisting of a path of length $n/3$ joined to a clique of size $2n/3$ gives the asymptotic upper bound for the cover time.

Cooper and Frieze [4] established the cover time of the giant component $C_1$ of the random graph $G_{n,p}, p = c/n$ where $c > 1$ is a constant. They showed in this setting that w.h.p. the cover time $C_{C_1}$ satisfies

$$C_{C_1} \approx \frac{cx(2 - x)}{4(cx - \ln c)} n(\ln n)^2,$$

where $x$ denotes the solution in $(0, 1)$ of $x = 1 - e^{-cx}$.

(Here $A_n \approx B_n$ if $A_n = (1 + o(1))B_n$ as $n \to \infty$.)

This raises the question as to what happens if $p = (1 + \varepsilon)/n$, $\varepsilon > 0$ and we allow $\varepsilon \to 0$. It is known that a unique giant component emerges w.h.p. only when $\varepsilon^3 n \to \infty$. Barlow, Ding, Nachmias and Peres [2] showed that w.h.p.

$$C_{C_1} = \Theta(n \log^2(\varepsilon^3 n)). \quad (1)$$

We prove in fact that

**Theorem 1.** Suppose that $N = \varepsilon^3 n \to \infty$ and $\varepsilon \to 0$. Then w.h.p.

$$C_{C_1} \approx n \log^2(\varepsilon^3 n).$$

This confirms a conjecture from [5], where it was shown that $C_{C_1^{(2)}} \approx \frac{\varepsilon}{4} n \log^2(\varepsilon^3 n)$ ($C_1^{(2)}$ is the 2-core of $C_1$, that is $C_1$ stripped of its attached trees). Our proof is very different from the proof in [5]. We will use the notion of a Gaussian Free Field (GFF). This was used in the breakthrough paper of Ding, Lee and Peres [9] that describes a *deterministic* algorithm for approximating $C_G$ to within a constant factor. This was later refined by Ding [10] and by Zhai [18]. It is the latter paper that we will use. In the next section, we will describe the tools needed for our proof. Then in Section 3 we will use these tools to prove Theorem 1.

## 2 Tools

### 2.1 Gaussian Free Field

For our purposes, given a graph $G = (V, E)$, a GFF is a centered normal vector $(\eta_v, v \in V)$ where

(i) $\mathbb{E}(\eta_v) = 0$ for all $v \in V$. 

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(ii) \( \eta_{\nu_0} = 0 \) is constant for some fixed vertex \( \nu_0 \in V \).

(iii) \( \mathbb{E}(\eta_v - \eta_w)^2) = R_{\text{eff}}(v, w) \) for all \( v, w \in V \).

Note that in particular, \( \text{Var}(\eta_v) = \mathbb{E}(\eta_v^2) = R_{\text{eff}}(v, \nu_0) \). (Here \( R_{\text{eff}} \) is the effective resistance between \( v \) and \( w \). See Doyle and Snell [11] or Lewin, Peres and Wilmer [16] for nice discussions of this notion.)

Next let

\[
M = \mathbb{E}(\max_{v \in V} \eta_v).
\]

Ding, Lee and Peres [9] proved that there are universal constants \( c_1, c_2 \) such that

\[
c_1 |E| M^2 \leq C_G \leq c_2 |E| M^2. \tag{2}
\]

Next let \( R = \max_{v, w \in V} R_{\text{eff}}(v, w) \), Zhai [18] proved that there are universal constants \( c_3, c_4 \) such that if we let \( \tau_{\text{cov}} \) be the first time that all the vertices in \( V \) have been visited at least once for the walk on \( G \) started at \( \nu_0 \), we have

\[
\mathbb{P}(\tau_{\text{cov}} - |E| M^2 \geq |E|((\sqrt{\lambda R} \cdot M + \lambda R)) \leq c_3 e^{-c_4 \lambda} \tag{3}
\]

for any \( \lambda \geq c_3 \). Setting \( X = \frac{\tau_{\text{cov}}}{|E| M^2} \), this gives after crude estimates

\[
|EX - 1| \leq E|X - 1| = \int_0^\infty \mathbb{P}(|X - 1| > t) dt \leq C \left( \sqrt{\frac{R}{M^2}} + \frac{R}{M^2} \right)
\]

for a universal constant \( C \). Since \( R \) and \( M \) do not depend on \( \nu_0 \), after taking the maximum over \( \nu_0 \) we thus get that \( C_G = \max_{\nu_0} \mathbb{E}\tau_{\text{cov}} \) satisfies

\[
C_G = |E| M^2 \left( 1 + O \left( \sqrt{\frac{R}{M^2}} + \frac{R}{M^2} \right) \right).
\]

Now, as we will see in the next section, the number of edges in the emerging giant satisfies

\[
|E| \approx 2\varepsilon n \quad \text{w.h.p.} \tag{4}
\]

We can therefore prove Theorem 1 by showing that in the case of the emerging giant we have w.h.p. that

\[
R = o(M^2) \quad \text{and} \quad M \approx \frac{\log(\varepsilon^3 n)}{(2\varepsilon)^{1/2}}. \tag{5}
\]

Now we know from (1), (2) and (4) that w.h.p. \( M = \Omega(\varepsilon^{-1/2} \log(\varepsilon^3 n)) \). Therefore to prove that \( R = o(M^2) \) it will be sufficient to prove

\[
R = O \left( \frac{\log(\varepsilon^3 n)}{\varepsilon} \right). \tag{6}
\]
### 2.2 Structure of the emerging giant

Ding, Kim, Lubetzky and Peres [7] describe the following construction of a random graph, which we denote by $H$. Let $0 < \mu < 1$ satisfy $\mu e^{-\mu} = (1 + \varepsilon)e^{-(1+\varepsilon)}$. Let $\mathcal{N}(\mu, \sigma^2)$ denote the normal distribution with mean $\mu$ and variance $\sigma^2$.

**Step 1.** Let $\Lambda \sim \mathcal{N}(1 + \varepsilon - \mu, \frac{1}{\varepsilon n})$ and assign i.i.d. variables $D_u \sim \text{Poisson}(\Lambda)$ ($u \in [n]$) to the vertices, conditioned that $\sum D_u 1_{D_u \geq 3}$ is even.

Let $N_k = |\{u : D_u = k\}|$ and $N_{\geq 3} = \sum_{k \geq 3} N_k$. Select a random graph $K_1$ on $N_{\geq 3}$ vertices, uniformly among all graphs with $N_k$ vertices of degree $k$ for $k \geq 3$.

**Step 2.** Replace the edges of $K_1$ by paths of lengths i.i.d. $\text{Geom}(1 - \mu)$ to create $K_2$. (Hereafter, $K_1$ denotes the subset of vertices of $H$ consisting of these original vertices of degree $\geq 3$ and $K_2 \supseteq K_1$ denotes the vertices created by the end of this step.)

**Step 3.** Attach an independent $\text{Poisson}(\mu)$-Galton-Watson tree to each vertex of $K_2$.

The main result of [7] is that for any graph property $A$, $\Pr(H \in A) \to 0$ implies that $\Pr(C_1 \in A) \to 0$, so we work with this construction for the remainder of the manuscript. For our application of the Gaussian free field, we make the convenient choice that $\nu_0$ is a vertex in $K_1$.

We next observe that

$$1 - \mu = \varepsilon + O(\varepsilon^2).$$

Applying the Chebyshev inequality we see that for any $\theta > 0$ we have

$$\Pr(|\Lambda - \mathbb{E}(\Lambda)| \geq \theta) \leq \frac{1}{\theta^2 \varepsilon n}.$$

Putting $\theta = N^{-1/3} \varepsilon$ (re-call that $N = \varepsilon^3 n$) we see that

$$\Lambda = 2\varepsilon + O(\varepsilon N^{-1/3} + \varepsilon^2), \quad \text{w.h.p.}$$

The restriction $\sum D_u 1_{D_u \geq 3}$ is even will be satisfied with constant probability and then we see that w.h.p.

$$N_{\geq 3} \approx \frac{4\varepsilon^3 n}{3} = \frac{4N}{3} \quad \text{and almost all vertices of } K_1 \text{ have degree three.}$$

The expected length of each path constructed by Step 2 will be asymptotically equal to $1/(1 - \mu) \approx 1/\varepsilon$. The path lengths are independent and so their sum will be concentrated around their mean which is asymptotically equal to $2\varepsilon^2 n$. Finally w.h.p. there will be no path longer than $2 \log N/\varepsilon$.

Furthermore, the expected size of each tree in Step 3 is also asymptotically equal to $1/\varepsilon$. These trees are independently constructed and so the total number of edges is concentrated around its mean which is asymptotically equal to $2\varepsilon n$. This justifies (4).
2.3 Normal Properties

In this section we describe several properties of the normal distribution that we will use in our proof.

First suppose that $g_1, g_2, \ldots, g_s$ are independent copies of $\mathcal{N}(0, 1)$. Then if $G_s = \max_{i=1}^s g_i$,

$$
\mathbb{E}(G_s) = \sqrt{2 \log s - \frac{\log \log s + \log(4\pi) - 2\gamma}{\sqrt{8 \log s}}} + O\left(\frac{1}{\log s}\right)
$$

(10)

where $\gamma = 0.577\ldots$ is the Euler-Mascheroni constant. For a proof see Cramér [6].

Next suppose that $(X_i)$ and $(Y_i)$ $1 \leq i \leq s$ are two centered Gaussian vectors in $\mathbb{R}^n$ such that $\mathbb{E}(X_i - X_j)^2 \leq \mathbb{E}(Y_i - Y_j)^2$ for all $1 \leq i, j \leq s$. Then,

$$
\mathbb{E}(\max \{X_i : i = 1, 2, \ldots, s\}) \leq \mathbb{E}(\max \{Y_i : i = 1, 2, \ldots, s\}).
$$

(11)

See Fernique [14], (Theorem 2.1.2 and Corollary 2.1.3). Finally we have that if $(X_i)_{1 \leq i \leq s}$ is a centered Gaussian vector and $\sigma^2 = \max_i \text{Var}(X_i)$, then

$$
\mathbb{E}(\max_{1 \leq i \leq s} X_i) \leq \sigma \sqrt{2 \log s}.
$$

(12)

This can be found, for example, in the appendix of the book by Chatterjee [3]; it follows from a simple union bound. Nevertheless, repeated carefully chosen applications of (12) will suffice to prove our upper bound on $M$. (Importantly, recall by comparison with (10) that independent normals are the asymptotically the worst case for the expected max.)

We also have

$$
\Pr(\left|\max_{1 \leq i \leq s} X_i - \mathbb{E}(\max_{1 \leq i \leq s} X_i)\right| > t) \leq 2e^{-t^2/2\sigma^2}.
$$

(13)

See for example Ledoux [15].

2.4 Galton-Watson Trees

A key parameter for us will be the probability that a Galton-Watson tree with Poisson($\mu$) offspring distribution survives for at least $k$ levels. The following Lemma was proved by Ding, Kim, Lubetzky and Peres (see Lemma 4.2 in [8]).

**Lemma 2.** Let $\mu$ be as in Section 2.2 and let $T$ be a Galton-Watson tree added in Step 3. Let $L_k$ denote the $k$-th level of $T$. For any $k \geq 1/\varepsilon$ we have

$$
\Pr(L_k \neq \emptyset) = \Theta(\varepsilon \exp \{-k(\varepsilon + O(\varepsilon^2))\}).
$$
Their proof also easily gives:

**Lemma 3.** For \( k < 1/\varepsilon \) we have

\[
\Pr(L_k \neq \emptyset) < \frac{10}{k}.
\]

It follows from Lemma 2 that the expected number of trees created in Step 3 of depth at least \( \gamma \varepsilon^{-1} \log N \), \( \gamma \geq 1/\log N \) lies between \( c_1 N \times \varepsilon^{-1} \times \varepsilon \exp\{-\gamma \log N + O(\varepsilon \log N)\} = c_1 N^{1-\gamma+O(\varepsilon)} \) and \( c_2 N^{1-\gamma+O(\varepsilon)} \) for some constants \( 0 < c_1 < c_2 \).

Conditioning on the results of Step 1 and Step 2, the number of such trees is distributed as a binomial with mean going to infinity and so we have that if \( 0 < \gamma < 1 \) then we have the following:

W.h.p. there are between \( \frac{1}{2}c_1 N^{1-\gamma+O(\varepsilon)} \) and \( 2c_2 N^{1-\gamma+O(\varepsilon)} \) trees of depth at least \( \gamma \varepsilon^{-1} \log N \).

(14)

The probability that any fixed tree has depth at least \( 2\varepsilon^{-1} \log N \) is \( O(\varepsilon N^{-2+o(1)}) \). There are w.h.p. \( O(\varepsilon^2 n) \) trees and so the expected number of trees with this or greater depth is \( O(\varepsilon^2 n \times \varepsilon N^{-(2+o(1))}) = O(N^{-(1+o(1))}) \). We therefore have the following.

W.h.p. there are no trees of depth exceeding \( \frac{2 \log N}{\varepsilon} \).

(15)

### 3 Proof of Theorem 1

#### 3.1 Effective resistance on the kernel

We begin by estimating the effective resistance between vertices of the *kernel* \( K_1 \). This is needed to justify (6).

We begin by shortening the induced paths between vertices created in Step 2 of Section 2.2. Let \( \ell_1 = \lceil 1/\varepsilon \rceil \). We first replace a path of length \( \ell \) by one of length \( \lceil \ell/\ell_1 \rceil \ell_1 \). Rayleigh’s Law ([11], [16]) implies that this increases all resistances between vertices. Let \( \hat{R}_{\text{eff}} \) denote the new resistances. Now every path has a length which is a multiple of \( \ell_1 \) and so if we replace paths, currently of length \( k\ell_1 \) by paths of length \( k \), then we change all resistances by the same factor \( \ell_1 \). So, if \( R^*_{\text{eff}} \) denotes these resistance then we have that

\[
R_{\text{eff}}(v, w) \leq \ell_1 R^*_{\text{eff}}(v, w) \text{ for all } v, w \in K_1. \tag{16}
\]

Let \( K^*_1 = (V^*, E^*) \) denote the graph obtained from \( K_1 \) in this way. Now we use the commute time identity ([11], [16]) for a random walk \( W^* \) on a graph \( K^*_1 \).

\[
R^*_{\text{eff}}(v, w)|E^*| = \tau(v, w) + \tau(w, v), \tag{17}
\]
where \( \tau(v, w) \) is the expected time for \( W^* \), started at \( v \) to reach \( w \).

Now the expected length of a path created in Step 2 of Section 2.2 is \( \approx 1/\varepsilon \) and so the expected length of a path created for \( K_1^* \) is at most 2. We then observe that if \( X \) denotes the length of a path created in Step 2 then

\[
\Pr(X \geq t) \leq (1 - (1 - o(1))\varepsilon)^t
\]

and so w.h.p. the union bound implies that no path is of length more than \( 2\varepsilon^{-1}\log N \) where \( N \) is as in (9). Because path lengths are independent, we see that w.h.p.

\[
2N \leq |E^*| \leq (1 + o(1)) \times 2N \times 2 \leq 5N.
\]

Now a simple argument based on conductance implies that w.h.p. the mixing time of \( W^* \) is \( \log^{O(1)} N \). Now for \( v, w \in V(K_1^*) \) we see that \( \tau(v, w) \) can be bounded by the mixing time plus the expected time to visit \( w \) from the steady state. The latter will be at most \( |E^*|/2 \) and so we see from (17) that

\[
\max \{ R_{\text{eff}}^*(v, w) : v, w \in K_1 \} = O(1).
\]

It then follows from (16) that

\[
\max \{ R_{\text{eff}}(v, w) : v, w \in K_1 \} = O(1/\varepsilon).
\]

Together with (15), this verifies (6).

From now on, we condition on \( C_1 \) having the required properties and work in the probability space defined by the GFF, with the one exception in equation (37).

### 3.2 Lower Bound

To prove Theorem 1 the main task is to determine the expected maximum \( \eta_v \). It turns out that for the lower bound, it suffices to consider the maximum over a very restricted set, consisting just of a single vertex from each sufficiently deep tree.

Consider the set of Galton-Watson trees of depth at least \( d = i\varepsilon^{-1}, i \) to be chosen, that are attached to a vertex within distance \( 1/\varepsilon \) of \( K_1 \) in \( G \). Choose one vertex at depth \( d \) from each tree to create \( S_d \). It follows from (14) with \( \gamma = i/\log N \), that there will be \( \approx cN^{1-\gamma+O(\varepsilon)} \) such trees for some constant \( c > 0 \). Let \( (\hat{\eta}_v)_{v \in S_d} \) be a random vector with i.i.d. \( N(0, \gamma \varepsilon^{-1} \log N) \) components. Then \( \hat{\eta}_v - \hat{\eta}_w \) has variance exactly \( 2\gamma\varepsilon^{-1}\log N \) whereas \( \eta_v - \eta_w \) has variance at least \( 2\gamma\varepsilon^{-1}\log N \) and so it follows from (11) that

\[
\mathbb{E}(\max \{ \eta_v : v \in S_d \}) \geq \mathbb{E}(\max \{ \hat{\eta}_v : v \in S_d \}).
\]

(19)
Applying (10) we see that
\[
E(\max \{ \hat{\eta}_v : v \in S_d \} ) \geq (1 + o(1))(2 \log(cN^{1-\gamma+O(\epsilon)})^{1/2} \times (\gamma \epsilon^{-1} \log N)^{1/2} \\
\approx \frac{(2\gamma(1-\gamma))^{1/2} \log N}{\epsilon^{1/2}}. \tag{20}
\]
Putting \( \gamma = 1/2 \) in (20) and applying (19) yields a lower bound for \( M = E(\max \{ \eta_v : v \in V \} ) \) sufficient for (5). It remains to determine a matching upper bound.

3.3 Upper Bound

We let \( \kappa \) denote the smallest power of 2 which is at least \( 1/\epsilon \), and will write \( \ell_0 = \log_2 \kappa \). We let \( L_k \) denote the set of vertices at distance \( k \) from \( K_2 \). We say that \( v \in G \) is a \( d \)-survivor if it has at least one \( d \)-descendant \( x_d(v) \); that is, a vertex \( x_d(v) \) such that \( \text{dist}(K_2, x_d(v)) = \text{dist}(K_2, v) + \text{dist}(v, x_d(v)) = \text{dist}(K_2, v) + d \).

Finally, we set \( U_0 = K_2 \) and define for each \( 1 \leq j \leq 2 \log N \) a set \( U_j \) by choosing, for each \( \kappa \)-survivor \( v \) in \( L_{(j-1)\kappa} \), an arbitrary \( \kappa \)-descendant \( x_{\kappa}(v) \). Evidently, we have for \( U = \bigcup_{j \geq 0} U_j \) that
\[
E(\max_{v \in V} \eta_v) \leq E(\max_{u \in U} \eta_u) + E(\max_{v \in V} (\eta_v - \eta_u(v))), \tag{21}
\]
for any function \( u : V \to U \). We will bound the two terms on the righthand side separately.

We begin with the first term. Let
\[
T_\delta = \frac{\epsilon \delta \log N}{(2 \epsilon)^{1/2}}
\]
where \( \delta = o(1) \) will be chosen below in (28). We then let \( Z_j = \max_{v \in U_j} \eta_v \)
and
\[
E(\max_{v \in U} \eta_v) = E\left( \max_{0 \leq j \leq 2 \log N} Z_j \right) \leq T_\delta + \sum_{j=0}^{2 \log N} \int_{t \geq T_\delta} \Pr(Z_j \geq t)dt. \tag{22}
\]
Now we have, where we write \( A \leq O B \) in place of \( A = O(B) \),
\[
E(|U_j|) \leq O(\epsilon^2 n \times (1 - \epsilon)^{\kappa(j-1)} \times \epsilon e^{-\epsilon \kappa} \leq N e^{-\epsilon \kappa j}, \quad j \geq 1. \tag{23}
\]

**Explanation:** We can assume that there are \( O(\epsilon^2 n) \) vertices that are roots of G-W trees i.e. are defined in Steps 1 and 2. Then the expected number of vertices at level \( \kappa(j-1) \) of a G-W tree will be \( (1 - \epsilon + O(\epsilon^2))^{\kappa(j-1)} = O((1 - \epsilon)^{\kappa(j-1)}) \). Then we use Lemma 2 to bound the number of \( \kappa \)-survivors.

**Case 1:** \( j \geq 1 \).

Now, assuming that the RHS of (23) grows faster than \( \log N \), we can assume that \( |U_j| \leq O \)
\( \frac{N}{e^{\varepsilon \kappa j}} \). Furthermore, if this expression is less than \( \log^2 N \) then we can use the Markov inequality to bound the size of \( |U_j| \) by \( \log^4 N \).

Now, if \( v \in U_j \) then \( \eta_v \) has variance \( \kappa_j + O(\varepsilon^{-1}) \). It then follows from Section 2.3 that

\[ E(Z_j) \leq (2\log(CN e^{-\varepsilon \kappa j} + \log^4 N))^{1/2} \times (\kappa_j + O(\varepsilon^{-1}))^{1/2}. \] (24)

\[ \Pr(Z_j \geq E(Z_j) + t) \leq 2 \exp \left\{ \frac{-t^2}{3\kappa \log N} \right\} \leq 2 \exp \left\{ \frac{-t^2}{3\kappa \log N} \right\}. \] (25)

Here \( C \) in (24) is a hidden constant from (23).

\[ \int_{t \geq T_\delta} \Pr(Z_j \geq t) dt \leq \int_{t \geq T_\delta} \exp \left\{ \frac{-(t-E(Z_j))^2}{3\kappa \log N} \right\} dt \leq \kappa^{1/2} \log^{1/2} N \exp \left\{ \frac{-(T_\delta - E(Z_j))^2}{3\kappa \log N} \right\}. \] (26)

Now if \( j \leq \frac{1}{100} \log N \) then (24) implies that \( E(Z_j) \leq (\kappa_j \log N)/9 \leq T_\delta/4 \) and similarly for \( \frac{99}{100} \log N \leq j \leq 2 \log N \). Otherwise, it follows from \( 2(xy)^{1/2} \leq x + y \) that we can write

\[ E(Z_j) \leq (2\varepsilon^{-1})^{1/2} \left( 1 + O \left( \frac{\log \log N}{\log N} \right) \right) (\varepsilon_j j)^{1/2} (\log N - \varepsilon \kappa j)^{1/2} \leq \left( 1 + O \left( \frac{\log \log N}{\log N} \right) \right) \log N \left( \frac{2\varepsilon}{\varepsilon_j j} \right)^{1/2} \leq e^{-\delta/2} T_\delta, \] (27)

if we take

\[ \delta = \frac{1}{\log^{1/3} N}. \] (28)

Plugging this into (26) we see that

\[ \int_{t \geq T_\delta} \Pr(Z_j \geq t) dt \leq \kappa^{1/2} \log^{1/2} N \times \log N^{-\Omega(\delta^2)} \leq N^{-\Omega(\delta^2)} T_\delta. \] (29)

Thus

\[ \sum_{j=1}^{2 \log N} \int_{t \geq T_\delta} \Pr(Z_j \geq t) dt \leq o(T_\delta). \] (30)

**Case 2:** \( j = 0 \).

It suffices to show that \( E(Z_0) = o(T_\delta) \) because then by (13),

\[ \int_{t = T_\delta}^{\infty} \Pr(Z_0 \geq t) dt \leq \int_{t = T_\delta}^{\infty} \exp \left\{ -\frac{(t - EZ_0)^2}{2(\frac{2}{\varepsilon} \log N + O(\varepsilon^{-1}))} \right\} dt = o \left( \sqrt{\frac{\log N}{\varepsilon}} \right) \] (31)
(by (18) and the fact that there are no paths longer than $\frac{2}{\varepsilon} \log N$, for every $v \in U^0$, $\eta_v$ has variance $\frac{2}{\varepsilon} \log N + O(\varepsilon^{-1}))$.

We have

$$E(Z_0) \leq E(\max_{v \in K_1} \eta_v) + E(\max_{u \in K_2} \min_{v \in K_1} \eta_u - \eta_v).$$

It follows from (18) that for $v_1, v_2 \in K_1$ we have $R_{\text{eff}}(u, v) \leq C/\varepsilon$ for some constant $C$. Thus by (12) and our choice that $v_0 \in K_1$ we have that

$$E\left(\max_{v \in K_1} \eta_v\right) \leq O \sqrt{2 \log(2N) \sqrt{C/\varepsilon}}. \quad (32)$$

To bound $E(\max_{v \in K_2} \min_{u \in K_1} \eta_u - \eta_v)$ we proceed as follows. We consider sets $I_0, I_1, I_2, \ldots$ of pairs of vertices from $K_2$ defined by the following rule:

For $v \in K_2$, if $2^i$ is the largest power of $2$ dividing $D = \text{dist}(v, K_1)$, then we add $(u, v)$ to $I_i$ for a single vertex $u$ lying at distance $2^i$ from $v$ and $D - 2^i$ from $K_1$. Notice that $I_0$ is simply the set of all edges of $K_2$.

Recall that $K_2$ has asymptotically $2\varepsilon^2 n$ vertices; thus we have w.h.p. that $|I_i| \leq 3\varepsilon^2 n/2^i$ for all $i$, say. In particular, assuming this bound (by conditioning that $C_1$ has this property) we have that

$$E\left(\max_{(v_1, v_2) \in I_i} \eta_{v_2} - \eta_{v_1}\right) \leq \sqrt{2^i} \sqrt{2 \log \left(\frac{3\varepsilon^2 n}{2^i}\right)}.$$  

Now, since each vertex $u \in K_2$ is joined to a vertex $v \in K_1$ by a path which uses at most one edge from each $I_i$, we can bound

$$E(\max_{u \in K_2} \min_{v \in K_1} \eta_u - \eta_v) \leq O \sum_{i=0}^{\log(2 \log N/\varepsilon)} \sqrt{2^i \log \left(\frac{3\varepsilon^2 n}{2^i}\right)}. \quad (33)$$

Here the upper limit of the sum comes from the fact that w.h.p. no induced path in $K_2$ is longer than $2 \log N/\varepsilon$. Notice that this is essentially a simple chaining argument (as in Dudley’s bound, see for instance [17]).

If $u_i$ is the summand in (33) then

$$\frac{u_{i+1}}{u_i} = 2^{1/2} \frac{\log(3\varepsilon^2 n) - (i + 1) \log 2}{\log(3\varepsilon^2 n) - i \log 2} = 2^{1/2} \left(1 - \frac{\log 2}{\log(3\varepsilon^2 n) - i \log 2}\right).$$

So, if $2^i \leq 3\varepsilon^2 n/100$ then $u_{i+1}/u_i \geq 4/3$. So, where $2^{i_0}$ is the largest power of $2$ that is less
than or equal to $3\varepsilon^2 n/100$ then

$$E\left(\max_{u \in K_2} \min_{v \in K_1} \eta_u - \eta_v\right) \leq O \sum_{i=i_0}^{\log(2 \log N/\varepsilon)} 2^i \log \left(\frac{3\varepsilon^2 n}{2^i}\right) \leq O \sum_{i=i_0}^{\log(2 \log N/\varepsilon)} 2^{i/2} \leq O \log^{1/2} N = o(\varepsilon^{1/2}) = o(T_\delta). \quad (34)$$

Combining (32) and (34) yields $E(Z_0) = o(T_\delta)$. Now it follows from (30) and (31) that

$$E(\max_{u \in U} \eta_u) \leq (1 + o(1))T_\delta. \quad (35)$$

Now let us bound the second term on the righthand side of (21). For this purpose we let $W_k = L_k \cup L_{2k} \cup L_{3k} \cup \ldots$ denote the set of vertices whose distance to $K_2$ is divisible by $k$. Our goal now is to show that a general vertex $v$ is close to some vertex $u \in U$ as measured by $(\eta_v - \eta_u)$; we will do this by showing that $v$ is close to its nearest (in graph distance) ancestor $y \in W_k$; this will suffice since our choice of $U$ ensures that some vertex $u \in U$ has the property that $y$ is also the closest ancestor of $u$ in $W_k$.

We will consider sets $J_0, J_1, J_2, \ldots, J_{\ell_0}$ of ordered pairs of vertices in $G$ with the following properties:

1. For $(v_1, v_2) \in J_i$, we have that $v_1, v_2 \in W_{2^i}$, and that $v_2$ is a $2^i$-descendant of $v_1$.
2. $J_0$ is the set of all edges in $G$ that are outside of $K_2$.
3. For each $i$, we have for each $2^i$-survivor $v_0 \in W_{2^i} \setminus W_{2^{i+1}}$ that exactly one $2^{i}$-descendant $x(v_0) \in W_{2^{i+1}}$ of $v_0$ is paired in $J_{i+1}$ with its $2^{i+1}$-ancestor $v_1 \in W_{2^{i+1}}$.
4. For all $i$, $\pi_2(J_{i+1}) \subset \pi_2(J_i)$. (Here $\pi_j$ is the projection function returning the $j$th coordinate of a tuple.)

Notice that pairings $J_0, J_1, \ldots, J_{\ell_0}$ with these properties exist by induction, and so we fix some choice of them. We write $\bar{J}_i$ for the set of unordered pairs which occur (in some order) in $J_i$.

The following simple observation is essential to our argument:

**Lemma 4.** Given any vertex in $v \in V$, whose closest ancestor in $W_\kappa$ is $\alpha(v)$, we have that there is a sequence $v = v_0, v_1, v_2, \ldots, v_t = \alpha(v)$ such that:

(a) For each $j = 1, \ldots, t$, $(v_{j-1}, v_j) \in \bar{J}_i$ for some $i$.

(b) For each $i = 0, \ldots, \ell_0$, at most $1 + 2(\ell_0 - i)$ of the pairs $(v_0, v_1), (v_1, v_2), \ldots, (v_{t-1}, v_t)$ belong to $\bar{J}_i$.  

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Proof. Given a vertex $v$, we define the parameters

$$
\phi(v) = \max \{0 \leq i \leq \ell_0 \mid v \in W_{2i}\}
$$

$$
\psi(v) = \max \{0 \leq i \leq \phi(v) \mid v \in \pi_2(J_i)\}.
$$

We claim that given any $v$, there is a vertex $a(v)$ such that either

(a) $\phi(a(v)) > \phi(v)$ and $(a(v), v) \in J_{\phi(v)}$, or else

(b) $\phi(a(v)) = \phi(v)$ and $\psi(a(v)) > \psi(v)$, and there exists $z(v)$ such that $(z(v), a(v))$ and $(z(v), v)$ are both in $J_{\psi(v)}$ for some $i$.

Observe that the Lemma follows from the claim; indeed, one can construct the claimed sequence recursively as follows: given the partially constructed sequence $v = v_0, v_1, \ldots, v_s$ we append either the single term $a(v_s)$ or the two terms $z(v_s), a(v_s)$, according to which case of part (a) of the claim applies, and terminate if $\phi(a(v_s)) = \ell_0$. Observe that a consecutive pair $v, v'$ in $v_0, \ldots, v_s$ only belongs (as an unordered pair) to $\tilde{J}_i$ only if either

(i) $v' = a(v)$ and $\phi(v') > \phi(v)$, or

(ii) $v' = z(v)$, the term after $v'$ is $v'' = a(v)$, and $\psi(v'') > \psi(v)$, or

(iii) the term before $v$ is $\hat{v}$, $v = z(\hat{v})$, $v' = a(\hat{v})$, and $\psi(v') > \psi(\hat{v})$.

Since $(\phi(v), \psi(v))$ increases lexicographically in this way along the path, we have the claimed upper bound of $1 + 2(\ell_0 - i)$ on the number of of consecutive pairs from $J_i$.

To prove the claim, consider the vertex $v$, and let $i = \phi(v)$. We consider two cases:

Case 1: $\psi(v) = \phi(v)$. In this case, by definition of $\psi(v)$, we have that there is a vertex $a(v)$ such that $(a(v), v) \in J_i$. In particular, as $2^i$ is the largest power of 2 in such that $v \in W_{2i}$ and $v$ is a $2^i$ descendant of $a(v)$, we have that $a(v) \in W_{2i+1}$; that is, that $\phi(a(v)) \geq i + 1$, as claimed.

Case 2: $\psi(v) = j < \phi(v)$. In this case, by definition of $\psi(v)$, we have that there is a vertex $z$ such that $(z, v) \in J_j$. Now by Property 3 of the pairings $\{J_i\}$, $z$ has a $2^j$-descendant $a(v)$ which is in $\pi_2(J_{j+1})$; in particular, we have that $\psi(a(v)) \geq j + 1 > \psi(v)$. (Note for clarity that $a(v)$ and $v$ are at the same distance from $K_1$ in Case 2 and so $\phi(a(v)) = \phi(v)$.) And by Property 4, $a(v) \in \pi_2(J_i)$ as well, and thus $(z, a(v)) \in J_i$, completing the proof of the claim.

Our next task is to bound $|J_i|$ for $0 \leq i \leq \ell_0$. We have from Property 3 and Lemma 3 that

$$
E|J_i| \leq O \ E|W_{2i}| \times \frac{1}{2^i} \leq O \sum_{j \geq 0} \frac{\varepsilon^2 n \mu^i}{2^i} \leq O \sum_{i \geq 0} \frac{\varepsilon^2 n}{2^i(1 - \mu^i)} \leq O \frac{\varepsilon n}{2^i}. \tag{36}
$$
It remains to show that the second term in (21) is \( o(T_\delta) \). Recall that given \( v \in V \), we choose \( u(v) \) to be a close vertex in \( U \) to \( v \) (in the graph distance). Without loss of generality we can assume that \( u(v) = \alpha(v) \), where \( \alpha(v) \) is provided by Lemma 4, because otherwise, since \( \alpha(u(v)) = \alpha(\alpha(v)) \), we write \( \eta_v - \eta_{u(v)} = (\eta_v - \eta_{\alpha(v)}) + (\eta_{\alpha(v)} - \eta_{\alpha(\alpha(v))}) + (\eta_{\alpha(u(v))} - \eta_{u(v)}) \) and by the triangle inequality we can obtain the same bound as below up to the constant 3. Thanks to Lemma 4, we decompose \( \eta_v - \eta_{\alpha(v)} = \sum_{j=1}^t \eta_{j-1} - \eta_j \) and using a chaining argument as before we get

\[
E_{H,\eta} \left( \max_{v \in V} |\eta_v - \eta_{\alpha(v)}| \right) \leq E_H \sum_{i=0}^{\ell_0} (1 + 2(\ell_0 - i))E_\eta \max_{(a,b) \in J_i} |\eta_a - \eta_b| \\
\leq O \sum_{i=0}^{\ell_0} (\ell_0 - i)\sqrt{2^i(\sqrt{2 \log |J_i|})}, \\
\leq O \sum_{i=0}^{\ell_0} (\ell_0 - i + 1)\sqrt{2^i(2 \log (\varepsilon n^{1/2}))}.
\]

Here, \( E_{H,\eta} \) is expectation over the larger space of the random graph \( H \) together with the GFF, while \( E_\eta \) is the expectation of a fixed Gaussian Free Field and \( E_H \) is an expectation just over the random choice of \( H \). In the last inequality we use (12) and Jensen’s inequality and the fact that \( \log^{1/2} x \) is a concave function. To get a high probability result, we will use the Markov inequality and this explains the \( \log^{1/4} N \) factor in (38) below. The last sum can essentially be dealt with as in (33). We check that the ratio between the terms \( i + 1 \) and \( i \) equals

\[
\frac{\ell_0 - i}{\ell_0 - i + 1} \sqrt{2^i \left(1 - \frac{2 \log 2}{\log(\varepsilon n) - 2i \log 2}\right)}
\]

which is strictly larger than, say \( \frac{10}{9} \) for \( 0 \leq i \leq \ell_0 - 10 \). Thus the last 10 terms dominate this sum and we get w.h.p.

\[
E_\eta \max_{v \in V} |\eta_v - \eta_{\alpha(v)}| \leq \log^{1/4} N \times \sqrt{2^\ell_0 \left(2 \log \left(\frac{\varepsilon n}{2^{2\ell_0}}\right)\right)} \leq O \frac{\log^{3/4} N}{\varepsilon^{1/2}} = o(T_\delta).
\]

References


