EDG E DISJOINT SPANNING TREES
IN RANDOM GRAPHS

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Abstract

We show that almost every $G_{m\text{-out}}$ contains $m$ edge disjoint spanning trees.

Introduction

In this note we consider the maximum number of edge disjoint spanning trees contained in the random graph $G_{m\text{-out}}$. Let a graph $G = (V, E)$ have property $A_k$ if it contains spanning trees $T_1, T_2, \ldots, T_k$ which are pair-wise edge disjoint.

We consider the random graph $G_m = G_{m\text{-out}}$. This has vertex set $V_n = \{1, 2, \ldots, n\}$. Each $v \in V_n$ independently chooses a set out $(v)$ of distinct vertices as neighbours, where each $m$-subset of $V_n \setminus \{v\}$ is equally likely to be chosen. This produces a random $m$ out-regular diagraph $D_m$ which has been selected uniformly from $\binom{n-1}{m}^n$ distinct possibilities. $G_m$ is obtained by ignoring orientation but without coalescing edges. (See [1], [2], [3] for properties of this model.)

Probability statements refer to the probability space of $D_m$ and graph theoretic statements refer to $G_m$.

**Theorem 1.** Let $m \geq 2$ be a fixed constant. Then

$$\lim_{n \to \infty} P(G_m \in A_m) = 1.$$ 

[This is clearly best possible.]

The major graph theoretic result underpinning our proof is as follows.

**Theorem 2** (Nash-Williams [5], Tutte [6]).

A graph $G = (V, E)$ has property $A_k$ if and only if for every partition $S_1, S_2, \ldots, S_t$ of $V$, $2 \leq t \leq |V|$, there at least $k(t - 1)$ edges of $G$ joining vertices in different subsets of the partition.

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(The necessity of the condition is obvious. The "meat" is in the sufficiency.)

**Proof** of main result. For \( S \subseteq V_n \) let \( \gamma(S) = |\{vw \in E(D_m): v \in S, w \notin S\}| \).

**Lemma 1.** The following events occur with probability tending to 1 (as \( n \to \infty \)).
(i) \( S \subseteq V_n, 1 \leq |S| \leq .49n \) implies \( \gamma(S) \geq m \)
(ii) \( S, T \subseteq V_n, S \cap T = \emptyset, |S|, |T| \geq .49n, \) implies \( \gamma(S) + \gamma(T) \geq m \).

**Proof.** Observe that \( \gamma(S) \geq |\{v \in S: \text{out}(v) \notin S\}|. \) Hence \( \gamma(S) \geq m \) for \( |S| \leq m \) and

\[
P(\exists S \subseteq V_n: m < |S| \leq .49n \text{ and } \gamma(S) < m) \leq \sum_{s = m + 1}^{.49n} \binom{n}{s} \binom{s}{m} \frac{1}{m} \frac{s - 1}{s - m + 1} \leq \sum_{s = m + 1}^{.49n} \binom{n}{s} s^{s - 1} \frac{s}{m} = \sum_s u_s, \text{ say.}
\]

Now

\[
\sum_{s = m + 1}^{[n^{1/3}]} u_s \leq \sum_{s = m + 1}^{[n^{1/3}]} \frac{n^s}{s} s^{s - 1} \frac{s}{m} \leq \sum_{s = m + 1}^{[n^{1/3}]} e^s s^{s - 1} \left( \frac{s}{m} \right) (s-m-1)(s-m) = O(n^{-m-1}m).
\]

Next let \( H(x) = a^x(1 - x)^{1-x} \), then

\[
\sum_{s = [n^{1/3}]}^{.49n} u_s \leq e^{o(n)} \sum_{s = [n^{1/3}]}^{.49n} e^{o(n)} H \left( \frac{s}{n} \right) \left( \frac{s}{n} \right)^m s \left( \frac{1}{n} \right) \left( 1 - \frac{s}{n} \right) \leq e^{o(n)} \sum_{s = [n^{1/3}]}^{.49n} \left( \frac{s}{n} \right)^m \left( 1 - \frac{s}{n} \right)^n = o(1),
\]

and (i) follows.

(ii)

\[
P(\exists S, T \subseteq V_n, |S|, |T| \geq .49n, S \cap T = \emptyset \text{ and } \gamma(S) + \gamma(T) < m) \leq \sum_{s = [.49n]}^{n-s} \sum_{t = .49n}^{s} \binom{n}{s} \binom{s-t}{t} \frac{s + t}{m(s + t - m + 1)} \leq n^2 2^{.51n^m - 1} n^{m-1} \gamma_{49} \gamma_{49} = o(1).
\]
Proof of Theorem 1. Let \( S_1, S_2, \ldots, S_t \) be a partition of \( V_n \) where \( |S_1| \geq |S_2| \geq \ldots \geq |S_t| \). Now in the graph \( G_m \), there precisely \( \gamma(S_1) + \gamma(S_2) + \ldots + \gamma(S_t) \) edges joining different subsets of the partition. But Lemma 1 implies

(ii) \( \gamma(S_1) + \gamma(S_2) \geq m \)

and

(i) \( \gamma(S_2) + \ldots + \gamma(S_t) \geq (t - 2)m \)

and so we can apply Theorem 3.

We note the following interesting consequence Theorem 1: \( G_{2-out} \) is super-eulerian with probability tending to one. (A graph is super-eulerian if it contains a trail which includes every vertex.) This is because every graph in \( A_2 \) has this property, Jaeger [4].

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References


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