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Edge-Colouring Random Graphs

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Let p_n denote the proportion of n-vertex graphs in class 2, that is, such that the chromatic index exceeds the maximum vertex degree. We extend a result of Erdős and Wilson, and show that $n^{-(1/2+\varepsilon)n} < p_n < n^{-(1/8-\varepsilon)n}$ for n sufficiently large.

1. INTRODUCTION

An edge-colouring of a (simple) graph G is an assignment of colours to the edges of G such that no two incident edges have the same colour. Thus the edges of any given colour form a matching. The edge-chromatic number or chromatic index $\chi' = \chi'(G)$ is the minimum number of colours in an edge-colouring of G. For a survey of results on edge-colouring see [8].

Of course $\chi' \geqslant \Delta$, where $\Delta = \Delta(G)$ is the maximum vertex degree of G. In 1964 Vizing [12] showed that always $\chi' = \Delta$ or $\Delta + 1$. Much attention has been focussed on finding conditions which imply that a graph is in class 1 ($\chi' = \Delta$) or in class 2 ($\chi' = \Delta + 1$).

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Vizing showed also that any graph in which the vertices of maximum degree induce an acyclic subgraph is in class 1. Erdős and Wilson [6] (see also Bollobás [1]) showed that the proportion of labelled graphs on n vertices with more than one vertex of maximum degree tends to 0 as $n \to \infty$. It follows that if p_n denotes the proportion of labelled graphs on n vertices which are in class 2, then $p_n \to 0$ as $n \to \infty$.

It is of interest to know how quickly $p_n \to 0$ as $n \to \infty$. The approach of Erdős and Wilson yields an upper bound on p_n of $O((n \log n)^{-1/2})$ at best (see Lemma 7 below). However, we shall see that for any $\varepsilon > 0$

$$n^{-(1/2+\varepsilon)n} < p_n < n^{-(1/8-\varepsilon)n}$$

for n sufficiently large.

This result will follow from the two theorems below on random graphs. Recall that $G_{n,p}$ is the random graph with vertex set $V_n = \{1, 2, ..., n\}$ in which the $\binom{n}{2}$ possible edges occur independently with probability p. We shall present a polynomial time algorithm A that attempts to Δ-edge-colour a graph. This algorithm runs in time $O(n^4)$ on a graph of order n (that is, with n vertices).

THEOREM 1. *Let p, c be constants, with $0 < p < 1$, $c < p/2$, $c < \frac{1}{3}$. Then*

$$P\{\text{algorithm } A \text{ fails to } \Delta\text{-edge-colour } G_{n,p}\} = 0(\exp\{-\tfrac{1}{2}cn \log n\}).$$

THEOREM 2. *Let p be a constant, with $0 < p < 1$. Then*

$$P\{G_{n,p} \text{ is in class } 2\} > \exp(-\tfrac{1}{2}n\{\log n + O(\log\log n)\}).$$

Let us recall some results concerning algorithms to edge-colour graphs. Holyer [10] showed that determining the edge-chromatic number of a graph is NP-hard, even for cubic graphs. Fournier [9] presented a polynomial time algorithm to edge-colour a graph G using at most $\Delta + 1$ colours. Further, this algorithm uses only Δ colours if the vertices of maximum degree in G induce an acyclic subgraph. Thus by the Erdős and Wilson result [6] we know that this algorithm will optimally edge-colour almost all graphs.

Given these two contrasting results it is natural to ask the question "Is there a polynomial expected time algorithm which optimally edge-colours all graphs?" By Theorem 1 it would suffice to find an algorithm that optimally edge-colours all graphs of order n in worst-case time $O(n^{\alpha n})$ for some $\alpha < \frac{1}{8}$. However, this still seems to be rather difficult.

In Section 2 below we introduce algorithm A together with a class of graphs on which it always works. Then in Section 3 we analyze this algorithm. Section 4 concerns lower bounds. In it we prove Theorem 2 and give lower bounds related to the Erdős–Wilson approach. Finally, in Section 5 we make a few concluding remarks.

2. A Colouring Algorithm

Recall from [9] that any graph whose vertices of maximum degree induce an acyclic subgraph can be Δ-edge-coloured, in $O(n^4)$ time. Given a graph G, denote by H the set of vertices of maximum degree in G. Our algorithm tries to remove a set S of matchings from G to obtain a subgraph G' such that (i) the set of vertices of maximum degree in G' is also H, (ii) H induces a stable set in G', and (iii) $\Delta(G') = \Delta(G) - |S|$. Thus, the vertices of maximum degree in G' form a stable set and G' can be $(\Delta(G) - |S|)$-edge-coloured. By also using each matching in S as a colour class we can colour G with $\Delta(G)$ colours, demonstrating graphically that G is in class 1.

It remains only to describe how to find our set S of matchings. Clearly the graph induced by the set H of vertices of maximum degree in G can be h-edge-coloured, where $h = |H|$. This produces a set of h matchings $M_1, ..., M_h$. In fact, we shall choose these matchings in an "inequitable" manner, so that the early matchings are large and the later matchings small. Finally, we shall form S by extending these matchings M_i to a set of disjoint matchings each covering most of G.

Before describing the algorithm in more detail, we describe a class of graphs on which it will always succeed. The graphs in which we are interested have two basic properties which allow us to extend our matchings as required. These are (i) the edges are distributed reasonably evenly throughout the graph, and (ii) the number of vertices of any given degree is not too large.

Definition. For $0 < p < 1$, $0 < c < 1$, and $0 < \varepsilon < \min\{p, 1-p\}$ we say that a graph G of order n is (p, c, ε)-*uniform* if

(i) $A \subseteq V(G)$, $|A| \geqslant \varepsilon n \Rightarrow \left| \dfrac{|E(A)|}{p\binom{|A|}{2}} - 1 \right| < \varepsilon$,

(ii) $A, B \subseteq V(G)$, $A \cap B = \phi$,

$$|A|, |B| \geqslant \varepsilon n \Rightarrow \left| \frac{|E(A, B)|}{p\,|A|\,|B|} - 1 \right| < \varepsilon,$$

(iii) the number of vertices of any given degree is at most cn.

Here $E(A)$ denotes the set of edges in G with both end vertices in A, and $E(A, B)$ denotes the set of edges in G with one end in A and the other in B.

LEMMA 1. *Let G be a (p, c, ε)-uniform graph of order n. Then, for n sufficiently large,*

$$\text{at most } \varepsilon n \text{ vertices have degree at most } \lceil (p-2\varepsilon)\, n \rceil \tag{1a}$$

and

$$\Delta(G) \geqslant (p-2\varepsilon)\, n + 1. \tag{1b}$$

Proof. We can assume that $\varepsilon < p/2$. If (1a) fails let A be a set of $s = \lceil \varepsilon n \rceil$ vertices of degree $\leqslant (p-2\varepsilon)\, n$ and $B = V(G) \backslash A$. Now

$$|E(A)| \geqslant (p-\varepsilon) \binom{s}{2} \qquad \text{by (i)},$$

and

$$|E(A, B)| \geqslant (p-\varepsilon)\, s(n-s) \qquad \text{by (ii)}.$$

However,

$$|E(A, B)| \leqslant \lceil (p-2\varepsilon)\, n \rceil\, s - 2\,|E(A)|,$$

and hence

$$\lceil (p-2\varepsilon)\, n \rceil\, s - 2(p-\varepsilon) \binom{s}{2} \geqslant (p-\varepsilon)\, s(n-s),$$

which is impossible for $n \geqslant (1+p)/\varepsilon - 1$. Equation (1b) follows directly from (1a).

We next show that the random graph $G_{n,p}$ is almost always (p, c, ε)-uniform.

LEMMA 2. *For fixed p, c, ε,*

$$P(G_{n,p} \text{ is not } (p, c, \varepsilon)\text{-uniform}) = O(n^{-(1-o(1))\, cn/2}).$$

Proof. We consider in order the three parts of the definition of (p, c, ε)-uniform.

(i) For $A \subseteq V_n$, $|E(A)|$ is distributed as the binomial random variable $B(\binom{|A|}{2}, p)$. Hence, using the Chernoff bound [3] we have

$$\begin{aligned} P(\text{(i) fails}) &\leqslant 2 \sum_{A : |A| \geqslant \epsilon n} e^{-(\varepsilon^2/6)(|A|-1)|A|p} \\ &= O(e^{-\varepsilon^4 n^2 p/7}). \end{aligned}$$

(ii) Since $|E(A, B)|$ is distributed as $B(|A||B|, p)$, a similar calculation gives $P(\text{(ii) fails}) = O(e^{-\varepsilon^4 n^2 p/4})$.

(iii) Let $a_n = \max\{\binom{n}{k} p^k(1-p)^{n-k}: 0 \leqslant k \leqslant n\}$. Then $a_n \sim (2\pi p(1-p)n)^{-1/2}$ (see Feller [7]). Let $k = \lceil cn \rceil$ and d satisfy $1 \leqslant k \leqslant n$ and $0 \leqslant d \leqslant n-1$. Use $d(v)$ to denote the degree of a vertex v. Then

$$P(d(1) = \cdots = d(k) = d) = \prod_{i=1}^{k} P(d(i) = d \mid d(1) = \cdots = d(i-1) = d)$$
$$\leqslant \prod_{i=1}^{k} a_{n-i}.$$

Hence

$$P(\text{(iii) fails}) \leqslant n2^n \prod_{i=1}^{k} a_{n-i} = O(n^{-(1-o(1))cn/2}).$$

We introduced (p, c, ε)-uniform graphs because our edge-colouring algorithm will be guaranteed to work on such graphs. We now turn to the algorithm itself. We shall use the following two well-known results in describing the algorithm for finding the set S of matchings and verifying that the algorithm works.

TUTTE'S THEOREM [11]. *A graph G has no perfect matching if and only if there is a set W of vertices in G such that the number of odd components of $G \backslash W$ is at least $|W| + 1$. If G has even order then the $|W| + 1$ here may be replaced by $|W| + 2$ (as was used, for example, in Erdős and Rényi [5]).*

EDMONDS'S ALGORITHM [4]. *There is a polynomial time ($O(n^3)$) algorithm which finds a perfect matching in any graph which has one.*

We now present in detail the procedure followed in finding the set S of matchings.

The algorithm

Input

A graph G of order n plus parameters p, ε.

begin

Step 1

{colours the edges induced by the vertices of maximum degree.}

$H :=$ the set of vertices of maximum degree in G;

$h := |H|$;

Edge-colour the subgraph induced by H using at most h colours, that is partition the edges contained in H into (possibly empty) matchings $M_1, M_2, \ldots, M_h$;

Furthermore we may assume {see Lemma 5} that this is done *inequitably* in the sense that

$$i \geqslant (p+\varepsilon)\,h \Rightarrow |M_i| \leqslant \theta n, \qquad \text{where } \theta = \varepsilon/\min\{p+\varepsilon, 1-p-\varepsilon\}; \quad (2)$$

Step 2
{extend $M_1, M_2, ..., M_h$ to cover all but a few vertices of low degree.}
begin
$G_0 := G$;
for $i = 1$ **to** h **do**
begin
Form G'_i from G_{i-1} by deleting the set $L_i \cup S_i$ of vertices, where L_i is the set of vertices covered by M_i and S_i is the set of vertices of degree less than $pn-(i-1)-2\varepsilon n$ in G_{i-1}. If necessary delete a vertex of $V(G_{i-1})\backslash(L_i \cup S_i)$ of minimum degree in G_{i-1} in order to make $|V(G'_i)|$ even;
Construct a perfect matching M'_i of G'_i (or fail);

$M^*_i := M_i \cup M'_i$; $G_i := G_{i-1}\backslash M^*_i$;

end
end;
Step 3
{tidying up}
Edge-colour G_h with $\Delta(G_h)$ colours (or fail);
together with the h matchings M^*_i, $i = 1, 2, ..., h$, this $\Delta(G)$-colours G.
end

3. Analysis of the Algorithm

We claim that the algorithm will succeed if

$$\text{each } G'_i \text{ has a perfect matching,} \quad (3a)$$

and

$$H \text{ is the set of vertices of maximum degree of } G_h. \quad (3b)$$

For if (3) holds then the set of vertices of maximum degree in G_h forms a stable set in G_h and so G_h can be $\Delta(G_h)$-edge-coloured. We will also have $\Delta(G) = h + \Delta(G_h)$ and so the algorithm will indeed succeed.

Lemma 3. *Let* $0 < p < 1$, $0 < c < \min\{p/2, \frac{1}{3}\}$ *be constants. If* $\varepsilon > 0$ *is a sufficiently small constant, n is sufficiently large, and G is a* (p, c, ε)*-uniform graph of order n then the algorithm succeeds in* Δ*-edge-colouring G.*

Proof. The main effort is to prove (3a). Once this is done, (3b) will follow easily.

Proof of (3a). Let $M_1, M_2, ..., M_h$ be an inequitable colouring of (the subgraph of G induced by) H-see Lemma 5 below.

Note first that if $v \in S_i$ then $d_G(v) < (p-2\varepsilon) n$. Thus if S is the union of the sets S_i then by Lemma 1 we have $|S| \leqslant \varepsilon n$ and $S \cap H = \phi$, if n is sufficiently large.

Suppose that the graph G_i' does not have a perfect matching. If an extra vertex v_i say was deleted when forming G_i' let $R_i = \{v_i\}$, and otherwise let $R_i = \phi$. Let $w = |W|$, where W is as in Tutte's theorem. Let $C_1, C_2, ..., C_k$, $k \geqslant w+2$, be the components of the graph $G_i' \backslash W$, where $|C_1| \leqslant |C_2| \leqslant \cdots \leqslant |C_k|$. We now have a partition $L_i, S_i, R_i, W, C_1, ..., C_k$ of the vertex set V_n of G. Then clearly

$$(w+2)|C_1| \leqslant k\,|C_1| \leqslant n. \tag{4}$$

By considering the degree, in G_{i-1}, of any vertex of C_1 we deduce that

$$w + |C_1| + |L_i| + |S_i| \geqslant pn - (i-1) - 2\varepsilon n$$

and hence

$$w + |C_1| \geqslant (p - 3\varepsilon)\, n - (i + |L_i|).$$

But inequitability implies that

$$i + |L_i| \leqslant (p + 1 + \varepsilon)\, h + 2\theta n$$

and so

$$\begin{aligned} w + |C_1| &\geqslant (p-3\varepsilon)\, n - hp - h - \varepsilon h - 2\theta n \\ &\geqslant (p - 3\varepsilon - p/3 - p/2 - \varepsilon - 2\theta)\, n \qquad \text{since } h \leqslant cn \\ &\geqslant pn/7 \qquad \text{for } \varepsilon \text{ sufficientlly small.} \end{aligned} \tag{5}$$

The proof now splits into two cases.

Case 1: $w \leqslant pn/14$.

The inequalities (4) and (5) now imply that $w \leqslant 14/p$ and hence $|C_1| \geqslant pn/8 \geqslant \varepsilon n$ for n sufficiently large and ε sufficiently small. Now let us consider vertex degrees in the graph G_{i-1}. For a vertex v and set U of vertices in G_{i-1} let $d(v, U)$ denote the number of edges in G_{i-1} between v and $U \backslash \{v\}$. Then

$$\sum_{x \in C_1} d_{G_{i-1}}(x) = \sum_{x \in C_1} d(x, W) + \sum_{x \in C_1} d(x, C_1) + \sum_{x \in C_1} d(x, L_i)$$
$$+ \sum_{x \in C_1} d(x, S_i) + \sum_{x \in C_1} d(x, R_i)$$
$$\leqslant \frac{14}{p} |C_1| + (p+\varepsilon)|C_1|^2$$
$$+ (p+\varepsilon)(|L_i| + \varepsilon n)|C_1| + \varepsilon n\, |C_1| + |C_1|$$

using uniformity and using $|L_i| + \varepsilon n$ to account for small L_i. Hence for large enough n

$$|C_1|((p-2\varepsilon)\, n - (i-1)) \leqslant |C_1|(p(|C_1| + |L_i|) + 3\varepsilon n)$$

and so

$$p\, |C_1| \geqslant pn - i - p\, |L_i| - 5\varepsilon n,$$

that is,

$$|C_1| \geqslant n - i/p - |L_i| - (5\varepsilon/p)\, n. \tag{6}$$

Subcase (a): $i \geqslant (p+\varepsilon)\, h$. Now $i \leqslant cn$ and by inequitability $|L_i| \leqslant 2\theta n$; and so by (6)

$$|C_1| \geqslant (1 - c/p)\, n - (5\varepsilon/p + 2\theta)\, n.$$

But this contradicts (4) for ε sufficiently small, since $1 - c/p > \frac{1}{2}$.

Subcase (b): $i < (p+\varepsilon)\, h$. Now $i \leqslant pcn + \varepsilon n$, and so by (6)

$$|C_1| \geqslant (1-c)\, n - |L_i| - (6\varepsilon/p)\, n.$$

Since $c < 1/3$, for ε sufficiently small this gives

$$|C_1| \geqslant 2n/3 - |L_i|.$$

But $k \geqslant 2$, and so

$$4n/3 - 2\, |L_i| \leqslant |C_1| + |C_2| \leqslant n - |L_i|.$$

This yields $|L_i| \geqslant n/3$, which contradicts $|L_i| \leqslant h$.

Case 2: $w > pn/14$.

Let $F = \bigcup \{C_j : |C_j| \leqslant 28/p\}$. Since $k \geqslant w + 2 \geqslant pn/14$, clearly $|F| \geqslant pn/28 \geqslant \varepsilon n$ for ε sufficiently small. Now

$$\sum_{x\in F} d_{G_{i-1}}(x) = \sum_{x\in F} d(x, F) + \sum_{x\in F} d(x, W) + \sum_{x\in F} d(x, L_i) + \sum_{x\in F} d(x, S_i) + \sum_{x\in F} d(x, R_i)$$
$$\leqslant \frac{28}{p}|F| + (p+\varepsilon)|W|\,|F| + (p+\varepsilon)|F|(|L_i| + \varepsilon n) + \varepsilon n\,|F| + |F|.$$

Hence for n sufficiently large

$$|F|((p-2\varepsilon)n - (i-1)) \leqslant |F|(p(|W| + |L_i|) + 3\varepsilon n)$$

and so

$$pw \geqslant pn - i - p\,|L_i| - 5\varepsilon n,$$

that is,

$$w \geqslant n - i/p - |L_i| - (5\varepsilon/p)\,n. \qquad (7)$$

This inequality is similar to (6) and will be used in a similar way.

Subcase (a): $i \geqslant (p+\varepsilon)\,h$. Now $i \leqslant cn$ and by inequitability $|L_i| \leqslant 2\theta n$; and so by (7)

$$w \geqslant (1 - c/p)\,n - (5\varepsilon/p + 2\theta)\,n$$
$$\geqslant n/2 \qquad \text{for } \varepsilon \text{ sufficiently small.}$$

But clearly $w < n/2$, a contradiction.

Subcase (b): $i < (p+\varepsilon)\,h$. Now $i \leqslant pcn + \varepsilon n$, and so by (7)

$$w \geqslant (1-c)\,n - |L_i| - (6\varepsilon/p)\,n.$$

But since $c < \frac{1}{3}$ we have $w \geqslant 2n/3 - |L_i|$ for ε sufficiently small. This yields

$$k > w \geqslant 2n/3 - |L_i| \geqslant n/3,$$

and also

$$n \geqslant w + k + |L_i| \geqslant 2n/3 + k,$$

a contradiction.

Proof of (3b). We note first that if $v \in H$ then for each $i = 1, 2, \ldots, h$, $d_{G_i}(v) = \Delta - i$, where $\Delta = \Delta(G)$. We saw earlier that $|S| \leqslant \varepsilon n$. Thus G will contain a set T of at least $n(1 - 2c - \varepsilon)$ vertices of degree at most $\Delta - 2$ in G, which are not in S. Note that $|T| \geqslant h$ for ε sufficiently small.

Suppose that (3b) fails. Then for some index i, $1 \leqslant i \leqslant h$, there must be a vertex v with $d_{G_{i-1}}(v) = \Delta - i$ which is missed by the matching M_i^*. Now for each vertex w in S_i we have $d_{G_{i-1}}(w) < pn - (i-1) - 2\varepsilon n \leqslant \Delta - i$. Hence v must be the one vertex in R_i. But this is not possible, since the number of vertices w in T with $d_{G_{i-1}}(w) < \Delta - i$ is at least

$$|T| - \left| \bigcup_{j<i} R_j \right| \geqslant |T| - (i-1) > 0,$$

and so some vertex in T would be chosen for R_i before v.

There is now only one step in our algorithm for edge-colouring which we have not described in detail. That is our method for inequitably colouring H if it is suitably large.

LEMMA 4. *Consider a graph H with order h. Given d with $0 \leqslant d \leqslant h$ set $B = \{x \mid d_H(x) \geqslant d\}$ and let $b = |B|$. If $b \leqslant d$ and $b \leqslant h - d$ then H can be h-edge-coloured using matchings $M_1, \ldots, M_h$ such that for $i > d$, $|M_i| \leqslant b$.*

Proof. By Vizing's theorem [12] we may assume that $B \neq \phi$. For each x in B choose a set E_x of $d_H(x) - (d-1)$ edges from x to $H \setminus B$. Let $F = \bigcup \{E_x : x \in B\}$. Let the graph H_1 be obtained from H by deleting the edges in F, and let the graph H_2 contain only the edges in F (and the same vertices as H). Then $\Delta(H_1) = d - 1$, and so H_1 can be d-edge-coloured using matchings M_1 to M_d. The graph H_2 is bipartite with stable sets B and $H \setminus B$ (where we are using H to denote also the vertex set of the graph H). The maximum degree in H_2 of a vertex in B is at most $h - 1 - (d-1) = h - d$. The maximum degree in H_2 of a vertex in $H \setminus B$ is at most b, which by assumption is at most $h - d$. Thus, $\Delta(H_2) \leqslant h - d$, and so H_2 can $(h-d)$-edge-coloured using matchings $M_{d+1}, \ldots, M_h$ (since H_2 is bipartite). Note that each of the matchings $M_{d+1}, \ldots, M_h$ uses at most b edges, as required.

LEMMA 5. *If a graph G of order n is (p, c, ε)-uniform and n is sufficiently large then the subgraph H can be inequitably edge-coloured.*

Proof. We may assume that $h > 2\theta n$. Let $d = (p + \varepsilon) h$, $B = \{x \in H : d_H(x) \geqslant d\}$, and $b = |B|$. We show first that $b < \varepsilon n$. Otherwise let $\hat{B} \subseteq B$ be of size $s = \lceil \varepsilon n \rceil$. Proceeding as in Lemma 1 we find, since $|H \setminus \hat{B}| > \varepsilon n$,

$$|E(\hat{B}, H \setminus \hat{B})| \leqslant (p + \varepsilon)\, s(h - s)$$

$$|E(\hat{B}, H \setminus \hat{B})| \geqslant (p + \varepsilon)\, hs - 2(p + \varepsilon) \binom{s}{2},$$

which yields a contradiction.

It is straightforward to check that $b \leqslant \min\{d, h-d\}$ since $h > 2\theta n$, and so we can apply Lemma 4 to establish Lemma 5.

The proof of Theorem 1 is now complete.

4. Lower Bounds

We now turn to the proof of Theorem 2. Clearly a regular graph of odd order is in class 2. We shall obtain our lower bound on the probability that a graph is in class 2 by counting regular graphs with degree about np. The main step in proving the lemma below is to show that for any n-vector $(d_1, \ldots, d_n)$ with $\sum_i d_i = nr$ there are no more graphs of order n with each degree $d(i) = d_i$ than with each $d(i) = r$.

Lemma 6. *Let p, α be constants, with $0 < p < 1$ and $\alpha \geqslant 1$. Let $r = r(n)$ be any integer such that $|r - np| \leqslant \alpha$ and rn is even. Then*

$$P(G_{n,p} \textit{ is r-regular}) = \exp\{-\tfrac{1}{2}n(\log n + O(\log\log n))\}.$$

Proof. Let $k = k(n) = 2(pqn \log n)^{1/2}$. Call a graph with n vertices *middling* if all its degrees lie in the range $(n-1)p \pm k$. By standard inequalities for binomial probabilities (see, for example, Feller [7] or Bollobás [2]),

$$P(G_{n,p} \text{ is middling}) = 1 - o(1/n).$$

Also (for given p), there is a constant $c > 0$ such that for all appropriate r (that is, integer r such that $|r - np| \leqslant \alpha$ and rn is even)

$$P(|E| = nr/2) > c/n.$$

Hence, if $A(n, r)$ denotes the set of graphs with n vertices which are middling and have exactly $nr/2$ edges, then

$$P(G_{n,p} \in A(n, r)) > c'/n \qquad \text{for some } c' > 0.$$

For each vector $\mathbf{x} = (x_1, \ldots, x_n)$ of non-negative integers let $f(\mathbf{x})$ be the number of graphs with vertex set $V_n = \{1, \ldots, n\}$ and with degree $d(i) = x_i$ for each $i = 1, \ldots, n$. Suppose that $x_1 \geqslant x_2 + 1$. Form $\mathbf{x}'$ by setting $x'_1 = x_1 - 1$, $x'_2 = x_2 + 1$, and $x'_i = x_i$ for $i = 3, \ldots, n$. We claim that

$$f(\mathbf{x}) \leqslant f(\mathbf{x}'). \tag{8}$$

To prove this, for each n-vector $\mathbf{y}$ of non-negative integers and graph H on $\{3, 4, \ldots, n\}$ let $f(\mathbf{y}, H)$ be the number of graphs with vertex set V_n and with $d(i) = y_i$ for each $i = 1, ..., n$ and such that the subgraph induced on $\{3, 4, ..., n\}$ is H. Thus

$$f(\mathbf{y}) = \sum_H f(\mathbf{y}, H),$$

where the sum is over all graphs H on $\{3, 4, ..., n\}$.

Consider a graph H such that $f(\mathbf{x}, H) > 0$. Let H have n_i vertices j of degree $x_j - i$ for $i = 0, 1, 2$. Then either (i) $n_1 + 2n_2 = x_1 + x_2$, in which case no graph counted by $f(\mathbf{x}, H)$ has vertices 1 and 2 adjacent, or (ii) $n_1 + 2n_2 = x_1 + x_2 - 2$, in which case each graph counted by $f(\mathbf{x}, H)$ has vertices 1 and 2 adjacent.

Now observe that if $a > b \geqslant 0$ then $\binom{a+b}{a} \leqslant \binom{a+b}{a-1}$. In case (i), $n_1 = (x_1 - n_2) + (x_2 - n_2)$, where $x_1 - n_2 > x_2 - n_2 \geqslant 0$. Thus

$$f(\mathbf{x}, H) = \binom{n_1}{x_1 - n_2} \leqslant \binom{n_1}{x_1' - n_2} = f(\mathbf{x}', H).$$

In case (ii), $n_1 = (x_1 - 1 - n_2) + (x_2 - 1 - n_2)$, where $x_1 - 1 - n_2 > x_2 - 1 - n_2 \geqslant 0$. Thus much as above

$$f(\mathbf{x}, H) = \binom{n_1}{x_1 - 1 - n_2} \leqslant \binom{n_1}{x_1' - 1 - n_2} = f(\mathbf{x}', H).$$

We have now shown that in either case $f(\mathbf{x}, H) \leqslant f(\mathbf{x}', H)$, and the claim (8) follows.

By (8) if $\sum_{i=1}^n x_i = nr$ then $f(\mathbf{x}) \leqslant f(\mathbf{r})$, where $\mathbf{r} = (r, r, ..., r)$. Thus

$$\begin{aligned} P(G_{n,p} &\text{ is } r\text{-regular} \mid G_{n,p} \in A(n, r)) \\ &= f(\mathbf{r})/|A(n, r)| \\ &\geqslant (2k+1)^{-n}. \end{aligned}$$

Hence

$$\begin{aligned} &P(G_{n,p} \text{ is } r\text{-regular}) \\ &\quad = P(G_{n,p} \text{ is } r\text{-regular} \mid G_{n,p} \in A(n, r))\, P(G_{n,p} \in A(n, r)) \\ &\quad \geqslant \exp\{-\tfrac{1}{2}n(\log n + \log\log n + O(1))\}. \end{aligned}$$

The other inequality is easy. As in the proof of Lemma 2 let a_n be the maximum probability mass of a binomial random variable $B(n, p)$. Then

$$\begin{aligned} P(G_{n,p} \text{ is } r\text{-regular}) &\leqslant \prod_{i=1}^{n-1} a_{n-i} \\ &< c^n(n!)^{-1/2} \qquad \text{for some constant } c>0 \\ &= \exp\{-\tfrac{1}{2}n(\log n + O(1))\}. \end{aligned}$$

Theorem 2 now follows easily, for a regular graph with an odd number of vertices is in class 2 (by counting edges). Thus if n is odd then Theorem 2 follows immediately from Lemma 6. For n even, consider the probability that the first vertex is isolated and the rest of the graph is regular.

We turn finally to the result of Erdős and Wilson [6]. Our last result shows that their approach cannot show that the probability that $G_{n,p}$ is in class 2 tends to zero very quickly as $n \to \infty$.

LEMMA 7. *Fix p, $0<p=1-q<1$. Let A be the event that a graph has two vertices of maximum degree, and let B be the event that the vertices of maximum degree induce a cycle. Then there are constants c, $c'>0$ such that*

$$P_n(A) > c(n \log n)^{-1/2}, \qquad P_n(B) > c'n^{-1} (\log n)^{-1/2}.$$

Proof. By a standard approximation to the binomial distribution (as in Lemma 6) there is a constant $c>0$ such that

$$\begin{aligned} P_n(d(v_1) = d(v_2) = pn+h) \\ > cn^{-1}\exp(-h^2/pqn) \end{aligned}$$

for all h, $0 \leqslant h \leqslant n^{2/3}/\log n$ say, such that $pn+h$ is an integer $<n$. (Indeed for n sufficiently large any $c<(2\pi pq)^{-1}$ will do.)

Hence, if $k=k(n)=(2pqn \log n)^{1/2}$ then

$$\begin{aligned} &P_n(d(v_1) = d(v_2) \geqslant pn+k) \\ &\qquad \geqslant c((n/\log n)^{1/2}-1)\, P(d(v_1)=d(v_2)=\lfloor pn+k+(n/\log n)^{1/2}\rfloor) \\ &\qquad \geqslant 2c'n^{-5/2}(\log n)^{-1/2} \qquad \text{for some constant } c'>0. \end{aligned}$$

(This probability is also $O(n^{-5/2}(\log n)^{-1/2})$.) We are using P_n to refer to the random graph $G_{n,p}$.

Now by a result of Bollobás [1],

$$P_n(\Delta < pn+k-2) \to 1 \qquad \text{as} \quad n \to \infty.$$

Let $d'(v_i)$ denote the number of edges incident with vertex v_i ignoring any edges to vertices v_1 or v_2. Then

$$\begin{aligned}P_n(d_1 = d_2 \geqslant pn + k > d_3, ..., d_n) \\ \geqslant P_n(d_1 = d_2 \geqslant pn + k \text{ and } d'_3, ..., d'_n < pn + k - 2) \\ = P_n(d_1 = d_2 \geqslant pn + k)\, P_{n-2}\{\Delta < pn + k - 2\} \\ \geqslant c'n^{-5/2} (\log n)^{-1/2}\end{aligned}$$

by the above, for n sufficiently large. Hence

$$\begin{aligned}P_n(A) \geqslant \binom{n}{2} c'n^{-5/2} (\log n)^{-1/2} \\ \geqslant c''(n \log n)^{-1/2} \qquad \text{for } n \text{ sufficiently large.}\end{aligned}$$

We may prove the result for $P_n(B)$ in just the same way. There is a constant $c > 0$ such that

$$\begin{aligned}P_n(d(v_1) = d(v_2) = d(v_3) = pn + h \text{ and } v_1, v_2, v_3 \text{ form a triangle}) \\ > cn^{-3/2} \exp(-3h^2/2pqn)\end{aligned}$$

for all h, $0 \leqslant h \leqslant n^{2/3}/\log n$ say, such that $pn + h$ is an integer $< n$. Hence much as above

$$\begin{aligned}P_n(d(v_1) = d(v_2) = d(v_3) \geqslant pn + k, \text{ and } v_1, v_2, v_3 \text{ form a triangle}) \\ > c'n^{-4}(\log n)^{-1/2}\end{aligned}$$

and we may complete the proof as before.

5. Concluding Remarks

(a) Let p_n denote the proportion of n-vertex graphs in class 2. We have shown that for any $\varepsilon > 0$

$$n^{-(1/2+\varepsilon)n} < p_n < n^{-(1/8-\varepsilon)n}$$

for n sufficiently large. It is natural to ask if there is a constant γ such that $p_n = n^{-(\gamma + o(1))n}$. If γ exists then of course $\frac{1}{8} \leqslant \gamma \leqslant \frac{1}{2}$. Can we tie it down further?

(b) If we are really interested in an algorithm to Δ-edge-colour graphs then it is clearly unsatisfactory to have to input as well as the graph G the extra parameters p and ε. It is not hard to remedy this.

Let algorithm A' be exactly like A, except that it has input only G and ε, and uses $\hat{p} = |E(G)|/\binom{n}{2}$ in place of p. This is the natural way to try to avoid having to input p. We may show that Theorem 1 holds with algorithm A replaced by A'. One way to see this involves tedious "uniform" versions of Lemmas 2 and 3 and the observation that $\hat{p}$ is close to p with very high probability.

Further it is easy to avoid having to input the parameter $\varepsilon > 0$. Either we can trace through the proofs to yield a specific upper bound for ε (for p in a suitable range for algorithm A'), or we may simply set $\varepsilon = \varepsilon(n) = 1/w(n)$, where $w(n) \geqslant 1$ is an arbitrarily chosen function such that $w(n) \to \infty$ as $n \to \infty$ and $w(n) = O(n^{1/5})$ say. For then by the proof of Lemma 2, the probability that part (i) or (ii) in the definition of (p, c, ε)-uniform fails is $O(\exp(-\varepsilon^4 n^2 p/7)) = O(\exp(-\delta n^{6/5}))$ for some $\delta > 0$.

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