Karp’s patching algorithm on dense digraphs

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June 18, 2020

Abstract

We consider the following question. We are given a dense digraph $D$ with minimum in- and out-degree at least $\alpha n$, where $\alpha > 1/2$ is a constant. The edges of $D$ are given edge costs $C(e), e \in E(D)$, where $C(e)$ is an independent copy of the uniform $[0,1]$ random variable $U$. Let $C(i,j), i,j \in [n]$ be the associated $n \times n$ cost matrix where $C(i,j) = \infty$ if $(i,j) \notin E(D)$. We show that w.h.p. the patching algorithm of Karp finds a tour for the asymmetric traveling salesperson problem that is asymptotically equal to that of the associated assignment problem. Karp’s algorithm runs in polynomial time.

1 Introduction

Let $\mathcal{D}(\alpha)$ be the set of digraphs with vertex set $[n]$ and with minimum in- and out-degree at least $\alpha n$. We are given a digraph $D \in \mathcal{D}(\alpha)$. The edges of $D$ are given independent edge costs $C(e), e \in E(D)$, where $C(e)$ is a copy of the uniform $[0,1]$ random variable $U$. Let $C(i,j), i,j \in [n]$ be the associated $n \times n$ cost matrix where $C(i,j) = \infty$ if $(i,j) \notin E(D)$. One is interested in the relationship between the optimal cost of the Assignment Problem (AP) and the Asymmetric Traveling Salesperson Problem (ATSP) associated with a cost matrix $A(i,j), i,j \in [n]$.

The problem AP is that of computing the minimum cost perfect matching in the complete bipartite graph $K_{n,n}$ when edge $(i,j)$ is given a cost $C(i,j)$. Equivalently, when translated to the complete digraph $\vec{K}_n$ it becomes the problem of finding the minimum cost collection of vertex disjoint directed cycles that cover all vertices. The problem ATSP is that of finding a single cycle of minimum cost that covers all vertices. As such it is always the case that $v(\text{ATSP}) \geq v(\text{AP})$ where $v(.)$ denote the optimal cost. Karp considered the case where $D = \vec{K}_n$. He showed that if the cost matrix comprised independent copies of $U$ then w.h.p. $v(\text{ATSP}) = (1 + o(1))v(\text{AP})$. He proves this by the analysis of a patching algorithm. Karp’s result has been refined in [2], [5] and [8].

Karp’s Patching Algorithm: First solve AP to create a collection $C_1, C_2, \ldots, C_\ell$ of vertex disjoint cycles covering $[n]$. Then patch two of the cycles together, as explained in the next paragraph. Repeat until there is one cycle.

∗Research supported in part by NSF grant DMS
Let $E(C)$ denote the set of edges in these cycles. A pair $e = (x, y), f = (u, v)$ in different cycles $C_1, C_2$ are said to be a patching pair if the edges $e' = (u, y), f' = (x, v)$ both exist. In which case we can replace $C_1, C_2$ by a single cycle $(C_1 \cup C_2 \cup \{(u, y), (x, v)\}) \setminus \{e, f\}$. The edges $e, f$ are chosen to minimise the increase in cost of the set of cycles.

**Theorem 1.** Suppose that $D \in D(\alpha)$ where $\alpha > 1/2$ is a positive constant. Suppose that each edge of $D$ is given an independent uniform $[0, 1]$ cost. Then w.h.p. $v(\text{ATSP}) = (1 + o(1))v(\text{AP})$ and Karp's patching algorithm finds a tour of the claimed cost in polynomial time.

## 2 Proof of Theorem 1

We begin by solving AP. We prove the following:

**Lemma 2.** W.h.p., after solving AP, the number $\nu_C$ of cycles $\ell \leq \ell_0 = n^{1/2} \log^3 n$.

**Lemma 3.** W.h.p., the solution to AP contains only edges of cost $A(i, j) \leq \alpha_0 = \frac{\log^4 n}{n}$.

Given these lemmas, the proof is straightforward. We can begin by replacing costs $X(e) > \alpha_0$ by infinite costs in order to solve AP. Lemma 3 implies that w.h.p. we get the same optimal assignment as we would without cost changes Having solved AP, the unused edges in $E(D)$ of cost greater than $\alpha_0$ have a cost which is uniform in $[\alpha_0, 1]$ and this is dominated by $\alpha_0 + U$.

Let $C = C_1, C_2, \ldots, C_m$ be a cycle cover and let $k_i = |C_i|$ where $k_1 \leq k_2 \leq \cdots \leq k_m, m \leq \ell_0$. If $e \in C_i$ then there at least $\varepsilon n - k_i$ edges $f$ such that $e, f$ make a patching pair. Different edges in $C_i$ give rise to disjoint patching pairs. We ignore the saving associated with deleting $e, f$ and only look at the extra cost $X(e') + X(f')$ incurred. We see that if $k_i < \varepsilon n / 2 \leq k_{i+1}$ then the number of possible patching pairs is at least

$$\phi(C) = \frac{1}{2} \sum_{i=1}^r k_i (\varepsilon n - k_i) \geq \frac{\varepsilon n}{2} \sum_{i=1}^r k_i \geq \frac{\varepsilon n r}{2} \geq \frac{\varepsilon n (m - 2\varepsilon^{-1})}{2}.$$

Let $\alpha_1 = \frac{1}{mn^{1/4}}$. Suppose we consider a sequence of patches where we always implement the cheapest patch. Let $E_i$ denote the event that the cost of the $i$th patch is at most $2(\alpha_0 + \alpha_1)$. We claim that as long as $m > 2\varepsilon^{-1}$ we have

$$P(E_i) \geq 1 - \ell_0 e^{-\varepsilon n^{1/4}/8} = o(n^{-1}). \quad (1)$$

Indeed,

$$P(\neg E_i \mid E_{i-1}) = P(\text{cheapest patch has cost greater than } 2(\alpha_0 + \alpha_1) \mid E_{i-1}) \leq \frac{(1 - \alpha_1^2)^{\varepsilon n (m - 2\varepsilon^{-1})}}{P(E_{i-1})} \leq \frac{P(E_{i-1})^{-1} \exp \left\{ -\frac{\varepsilon n (m - 2\varepsilon^{-1})}{m^{2/14}} \right\}}{P(E_{i-1})^{-1} e^{-\varepsilon n^{1/4}/8}}. \quad (2)$$

This implies that

$$P(\neg E_i) \leq e^{-\varepsilon n^{1/4}/8} + P(\neg E_{i-1}).$$

Given that $P(E_0) = 1$, this gives an inductive proof of (1).

When $m \leq 2\varepsilon^{-1}$, we have

$$\phi(C) \geq \varepsilon n \quad \text{and} \quad P(\neg E_i \mid E_{i-1}) \leq P(E_{i-1})^{-1} e^{-\varepsilon n a_2^2} = P(E_{i-1})^{-1} e^{-\varepsilon n^{3/4}} \quad (3)$$
and we can proceed as for \( m > 2\varepsilon^{-1} \).

It follows from \([2]\) and \([3]\) and the union bound that w.h.p.

\[
v(ATSP) \leq v(AP) + 2 \sum_{m=1}^{\ell_0} \left( \frac{\log^4 n}{n} + \frac{1}{mn^{1/8}} \right) = (1 + o(1))v(AP).
\]

The last equality follows from the fact that w.h.p. \( v(AP) > (1 - o(1))\zeta(2) > 1 \) where the lower bound of \((1 - o(1))\zeta(2)\) comes from \([1]\).

### 3 Proof of Lemma 2

Let \( G \) denote the bipartite graph with vertex partition \( A = \{a_1, a_2, \ldots, a_n\}, B = \{b_1, b_2, \ldots, b_m\} \) and an edge \( \{a_i, b_j\} \) for every directed edge \((i, j) \in D\). We let \( A_r = \{a_1, a_2, \ldots, a_r\} \) and we let \( M_r \) be the minimum cost matching of \( A_r \) into \( B \) and let \( B_r \) be the \( B \)-endpoints of the edges in \( M_r \). We obtain \( M_r \) from \( M_{r-1} \) by finding an augmenting path \( P = (a_1, \ldots, a_\sigma, b_{\phi(\sigma)}) \) from \( a_r \) to \( B \setminus B_{r-1} \) of minimum additional weight. So, in this notation, \( M_r \) matches \( A_r \) with \( \{b_{\phi(i)}, i = 1, 2, \ldots, r\} \).

The matching \( M_{r-1} \) induces a collection \( C_{r-1} \) of vertex disjoint paths and cycles in \( D \). An augmenting path \( P \) with respect to \( M_{r-1} \) changes this collection in the following way. We first add an edge \( \{a_r, b_k\} \) for some index \( k \). If \( b_k \) is isolated in \( M_r \) then \( C_r = C_{r-1} \) plus the edge \((r, k)\). Otherwise, suppose that \((a_s, b_k) \in M_r \). This means that adding the edge \((r, k)\) to \( C_r \) increases the in-degree of \( k \) to two. So, we delete the edge \((s, k)\) and then continue along \( P \) to examine the other edge \((a_s, b_l)\) incident with \( a_s \) in the path. This continues until we reach \( a_\sigma \). We then add the edge \((a_\sigma, b_{\phi(\sigma)})\). In the digraph \( D \) this either means that the added edge closes a cycle or connects two paths into one.

Suppose now that \( a_\sigma \) has \( \delta_r \) neighbors in \( B \setminus B_{r-1} \). Conditional on the history of the algorithm, each of these \( \delta_r \) vertices is equally likely to be \( b_{\phi(\sigma)} \). It follows that the probability the edge \((\sigma, \phi(\sigma))\) closes a cycle is exactly \( 1/\delta_r \). It follows that

\[
\mathbb{E}(\nu_r) \leq \mathbb{E}\left( \sum_{r=1}^{n} \frac{1}{\delta_r} \right).
\]

We will prove below in Section 4 that

\[
\mathbb{P}(\delta_r \leq \delta) \leq \frac{\delta \nu_1}{\theta_r},
\]

where

\[
\nu_1 = 2 \log^4 n \quad \text{and} \quad \theta_r = \min\{\alpha n, n-\rho\}.
\]

It follows that for any choice of \( \gamma_r \) we have

\[
\mathbb{E}\left( \frac{1}{\delta} \right) \leq \sum_{r=1}^{\gamma_r} \frac{\mathbb{P}(\delta_r \leq \delta)}{\delta} + \frac{1}{\gamma_r} \leq \frac{\gamma_r \nu_1}{\theta_r} + \frac{1}{\gamma_r}.
\]

The best choice for \( \gamma_r \) is \( (\theta_r/\nu_1)^{1/2} \) from which we deduce that

\[
\mathbb{E}(\nu_r) \leq 2 \sum_{r=1}^{(1-\alpha)n} \frac{\nu_1^{1/2}}{(\alpha n)^{1/2}} + 2 \sum_{r=(1-\alpha)n}^{n-1} \frac{\nu_1^{1/2}}{(n-r)^{1/2}} \leq 4 \left( \frac{\nu_1 n \alpha}{\alpha} \right)^{1/2}.
\]

Lemma 2 follows from the Markov inequality.
4 Proof of Lemma 3

Chernoff Bounds: We use the following inequalities associated with the Binomial random variable $Bin(n, p)$.

$$\mathbb{P}(Bin(n, p) \leq (1 - \varepsilon)np) \leq e^{-\varepsilon^2 np/2}.$$  
$$\mathbb{P}(Bin(n, p) \geq (1 + \varepsilon)np) \leq e^{-\varepsilon^2 np/3}.$$  
$$\text{for } 0 \leq \varepsilon \leq 1.$$  
$$\mathbb{P}(Bin(n, p) \geq \gamma np) \leq \left(\frac{e}{\gamma}\right)^{\gamma np}.$$  
$$\text{for } \gamma \geq 1.$$  

Proofs of these inequalities are readily accessible, see for example [4].

We use the notation

Let

$$\mu = \frac{n}{\log n}; \quad w_0 = \frac{1}{\mu}; \quad w_1 = w_0 \log n = \frac{\log^4 n}{n}.$$  

The aim of this section is to show that w.h.p. no edges of weight more than $w_1$ are used in the construction of $M_n$. For a set $S \subseteq A$ we let

$$N_0(S) = \{b \in B : (a, b) \in E(G) \text{ and } w(a, b) \leq w_0 \text{ for some } a \in S\}.$$  

Let

$$\beta = \frac{\alpha}{2} + \frac{1}{4} \text{ and } \gamma = 3(\log \beta^{-1} + 1).$$

and let

$$r_1 = \frac{\mu}{10}; \quad r_2 = \frac{\alpha n}{1000}; \quad r_3 = \beta n; \quad r_4 = n - \frac{n}{\gamma \log^2 n}; \quad r_5 = n - \log^2 n.$$  

Lemma 4. W.h.p. we have

$$|N_0(S)| \geq \frac{\alpha n |S|}{3\mu} \text{ for all } S \subseteq A, 1 \leq |S| \leq r_1.$$  
$$|N_0(S)| \geq \frac{n}{40} \text{ for all } S \subseteq A, r_1 < |S| \leq r_2.$$  
$$|N_0(S)| \geq \beta n + 1 \text{ for all } S \subseteq A, r_2 < |S| \leq r_3.$$  
$$|N_0(S)| \geq r_4 \text{ for all } S \subseteq A, |S| > r_3.$$  
$$n - |N_0(S)| \leq \frac{n - |S|}{\log n} \text{ for all } S \subseteq A, |S| \geq r_4.$$  

Proof. We first observe that for a fixed $S \subseteq A, 1 \leq |S| \leq \beta n$ we have $|N_0(S)|$ stochastically dominates $Bin(\alpha n, q)$ where $q = 1 - (1 - w_0)^s$. If $1 \leq |S| \leq r_1$ then $q \geq s/2\mu$. So,

$$\mathbb{P}(\neg(\text{7})) \leq \sum_{s=1}^{r_1} \binom{n}{s} \mathbb{P}(Bin(\alpha n, q) \leq \frac{\alpha ns}{3\mu}) \leq \sum_{s=r_1}^{r_2} \binom{nc}{s} \left(\frac{e}{s}\right)^{s} e^{-sn/20\mu} = o(1).$$

If $r_1 < |S| \leq r_2$ then $q > 1/20$. So,

$$\mathbb{P}(\neg(\text{8})) \leq \sum_{s=r_1}^{r_2} \binom{n}{s} \mathbb{P}(Bin \left(\alpha n, \frac{1}{20}\right) \leq \frac{an}{40}) \leq \left(\frac{ane}{r_2}\right)^{r_2} e^{-an/80} = o(1).$$
If \( r_2 < |S| \leq r_3 \) then \( q \geq 1 - e^{-\alpha n / 2000\mu} \). So,

\[
\mathbb{P}(\neg(9)) \leq \mathbb{P}(\text{Bin}(\alpha n, q) \leq \beta n) \leq \left(\frac{\alpha n}{\beta n}\right)^{(1 - q)(\alpha - \beta)n} \leq 2^n e^{-\alpha(\alpha - \beta)n^2 / 2000\mu} = o(1).
\]

For (10) let \( T, |T| = t = \frac{n}{\gamma \log n} \) denote a set of vertices with no neighbours in \( S \). Each member of \( B \) has at least \( 3\varepsilon n / 2 \) \( G \)-neighbors in \( S \). Thus,

\[
\mathbb{P}(\neg(10)) \leq \sum_{s=r_3}^{n} \binom{n}{s} \left(\frac{n}{t}\right)^{1 - w_0} (1 - w_0)^{3\varepsilon n / 2} \leq 2^{n + o(n)} e^{-\Omega(n \log n)} = o(1).
\]

For (11) let \( T \) play the same role but with \( t = |T| = \frac{n - s}{\log n} \). Then,

\[
\mathbb{P}(\neg(11)) \leq \sum_{s=r_4}^{r_5} \binom{n}{s} \left(\frac{n}{t}\right)^{1 - w_0} (1 - w_0)^{3\varepsilon n / 2} \leq \sum_{s=r_4}^{r_5} \left(\frac{ne}{n - s} \cdot \left(\frac{ne \log n}{n - s}\right)^{1/\log n}\right) \cdot \exp \left\{-\frac{3\varepsilon \log^2 n}{2}\right\} \right)^{n-s} = o(1).
\]

Lemma 5. W.h.p., no edge of length at least \( w_1 \) appears in any \( M_r, r \leq n \).

Proof. We first consider \( 1 \leq r \leq r_1 \). Choose \( a \in A_r \) and let \( S_0 = \{a\} \) and let an alternating path \( P = (a = u_1, v_1, \ldots, u_{k-1}, v_k, \ldots) \) be acceptable if \( (i) \ u_1, \ldots, u_{k}, \ldots \in A, \ v_1, \ldots, v_{k-1}, \ldots \in B, \ (ii) \ u_{i+1}, v_i \in M_r, i = 1, 2, \ldots \) and \( (iii) \ w(u_i, v_i) \leq w_0, i = 1, 2, \ldots \)

Now consider the sequence of sets \( S_0 = \{a_0\}, S_1, S_2, \ldots, S_i, \ldots \) defined as follows:

Case (a): \( N_0(S_i) \subseteq \phi(A_r) \). In this case we define \( S_{i+1} = \phi_r^{-1}(T_i) \), where \( T_i = N_0(S_i) \). By construction then, every vertex in \( S_j, j \leq i + 1 \) is the endpoint of some acceptable alternating path.

Case (b): \( T_i \setminus \phi(A_r) \neq \emptyset \). In this case there exists \( b \in T_i \) which is the endpoint of some acceptable augmenting path.

It follows from (7) applied to \( S_i \) that w.h.p. there exists \( k = o(\log n) \) such that \( |N_0(S_k)| > r \) and so Case (b) holds. This implies that if \( 1 \leq r \leq r_1 \) then \( w(a, \phi_r(a)) \leq kw_0 \) for all \( a \in A_r \). For if \( w(a, \phi_r(a)) > kw_0 \) then there are at least \( \Omega(rn / \mu) \) choices of \( b \in B \setminus \phi(A_r) \) such that we can reduce the matching cost by deleting \( (a, \phi_r(a)) \) and changing \( M_r \) via an acceptable augmenting path from \( a \) to \( b \). The extra cost of the edges added in this path is \( o(w_0 \log n) \).

Now consider \( r_1 < r \leq r_2 \). We know that w.h.p. there is \( k = o(\log n) \) such that \( |S_k| > r_1 \) and that by (8) we have that w.h.p. \( |N_0(S_{k+1})| > n / 40 > r \) and we are in Case (b) and there is a low cost augmenting path for every \( a, \) as in the previous case. When \( r_2 < |S_k| \leq r_3 \) we use the same argument and find by (9) we have w.h.p. \( N_0(S_{k+1}) > r_3 \geq r \) and there is a low cost augmenting path. Similarly for \( r_3 < r \leq r_4 \), using (10) and for \( r_4 < r \leq r_5 \) using (11), \( o(\log n) \) times. When \( r > r_5 \) we can use the fact that w.h.p. for every vertex \( b \in B \) has at least \( (1 - o(1)) \log^3 n \) vertices \( a \in A \) such that \( w(a, b) \leq w_0 \).

Finally note that the number of edges in the augmenting paths we find is always \( o(\log n) \).

Now,

\[
\mathbb{P}(\exists a \in A : \{|e : v \in e, X_e \leq w_1| \geq \nu_1\}) \leq \mathbb{P}\left(\text{Bin} \left(n, \frac{\log^4 n}{n}\right) \geq \nu_1\right) = O(n^{-2}). \tag{12}
\]
Let $\zeta_a$ denote the number of times that vertex $a$ takes the role of $a_\sigma$. It follows from (12) that w.h.p.

$$\zeta_a \leq \nu_1, \text{ for all } a \in A.$$  \hspace{1cm} (13)

We now use (13) to prove (5). Consider now how a vertex $a \in A$ loses neighbors in $B \setminus B_r$. It can lose up to $\nu_1$ for the times when $a = a_\sigma$. Otherwise, it loses a neighbor when $a_\sigma \neq a$ chooses a common neighbor with $a$. The important point here is that this choice depends on the structure of $G$, but not on the weights of edges incident with $a$. It follows that the cheapest neighbors at any time are randomly distributed among the current set of available neighbors. To get to the point where $a_\sigma = a$ and $\delta_r \leq \delta$, we must have at least one of the $\nu_1$ original cheapest neighbors occurring in a random $\delta$ subset of a set of size at least $\theta_r = \min \{\alpha n, n - r\}$. The probability of this is $\nu_1 \delta / \theta_r$.

5 Final Remarks

We have extended the proof of the validity of Karp’s patching algorithm to dense graphs with minimum in- and out-degree at least $\alpha n$ and uniform $[0, 1]$ edge weights. It is a routine exercise to extend the analysis to costs with a distribution function $F(x)$ that satisfies $F(x)/x \searrow 1$ as $x \to 0$. Janson [6] describes a nice simple coupling in the case of shortest paths. Our argument fails if $\alpha \leq 1/2$. In this case the assignment problem may not be feasible. One can consider adding randomly weighted random edges as in Frieze [3], but there seems to be a technical difficulty at present in extending the results in this direction.

References


