On the insertion time of random walk cuckoo hashing

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Abstract

We show that if the number of hash functions \( d = O(1) \) is sufficiently large, then the expected insertion time of Random Walk Cuckoo Hashing is \( O(1) \) per item.

1 Introduction

Our motivation for this paper comes from Cuckoo Hashing (Pagh and Rodler [8]). Briefly each one of \( n \) items \( x \in L \) has \( d \) possible locations \( h_1(x), h_2(x), \ldots, h_d(x) \in R \), where \( d \) is typically a small constant and the \( h_i \) are hash functions, typically assumed to behave as independent fully random hash functions. (See [7] for some justification of this assumption.)

We assume each location can hold only one item. Items are inserted consecutively and when an item \( x \) is inserted into the table, it can be placed immediately if one of its \( d \) locations is currently empty. If not, one of the items in its \( d \) locations must be displaced and moved to another of its \( d \) choices to make room for \( x \). This item in turn may need to displace another item out of one of its \( d \) locations. Inserting an item may require a sequence of moves, each maintaining the invariant that each item remains in one of its \( d \) potential locations, until no further evictions are needed.

We now give the formal description of the mathematical model that we use. We are given two disjoint sets \( L = \{v_1, v_2, \ldots, v_n\} \), \( R = \{w_1, w_2, \ldots, w_m\} \). Each \( v \in L \) independently chooses a set \( N(v) \) of \( d \geq 2 \) random neighbors in \( R \). This provides us with the bipartite cuckoo graph \( \Gamma \). Cuckoo Hashing can be thought of as a simple algorithm for finding a matching \( M \) of \( L \) into \( R \) in \( \Gamma \). In the context of hashing, if \( \{x, y\} \) is an edge of \( M \) then \( y \in R \) is the hash value of \( x \in L \).

Cuckoo Hashing constructs \( M \) by defining a sequence of matchings \( M_1, M_2, \ldots, M_n \), where \( M_k \) is a matching of \( L_k = \{v_1, v_2, \ldots, v_k\} \) into \( R \). We let \( R_k \) denote the vertices of \( R \) that are covered by \( M_k \) and define the function \( \phi_k : L_k \to R_k \) by asserting that \( M_k = \{\{v, \phi_k(v)\} : v \in L_k\} \). We obtain \( M_k \) from \( M_{k-1} \) by finding an augmenting path \( P_k \) in \( \Gamma \) from \( v_k \) to a vertex in \( \bar{R}_{k-1} = R \setminus R_{k-1} \).

This augmenting path \( P_k \) is obtained by a random walk. To begin we obtain \( M_1 \) by letting \( \phi_1(v_1) \) be a random member of \( N(v_1) \). Having defined \( M_k \) we proceed as follows: Steps 1 – 4 constitute round \( k \).

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Algorithm insert:

Step 1 \( x \leftarrow v_k; M \leftarrow M_{k-1}; \)

Step 2 If \( S_k(x) = N(x) \cap \bar{R}_{k-1} \neq \emptyset \) then choose \( y \) randomly from \( S_k(x) \) and let \( M_k = M \cup \{x,y\} \), else

Step 3 Choose \( y \) randomly from \( N(x) \);

Step 4 \( M \leftarrow M \cup \{x,y\} \setminus \{y, \phi_k^{-1}(y)\}; x \leftarrow \phi_k^{-1}(y); \textbf{goto} \) Step 2.

Our interest here is in the expected time for insert to complete a round. For large \( d \), we will improve on the results of Frieze, Melsted and Mitzenmacher [5], Fountoulakis, Panagiotou and Steger [2], Fotakis, Pagh, Sanders and Spirakis [3]. Mitzenmacher [6] gives a survey on Cuckoo Hashing and Frieze and Melsted [4], Fountoulakis and Panagiotou [1] give information on the relative sizes of \( L, R \) needed for there to exist a matching of \( L \) into \( R \).

We will prove the following theorem:

**Theorem 1** Suppose that \( n = (1 - \varepsilon)m \) where \( \varepsilon \) is a fixed positive constant. Then there exists a constant \( d_\varepsilon \) such that if \( d \geq d_\varepsilon \) then w.h.p. \( \Gamma \) is such that over the random choices in Steps 2,3,$$
\mathbb{E}(|P_k|) \leq 2 \text{ for } k = 1, 2, \ldots, n.
$$ (1)

Here \( |P_k| \) is the length (number of edges) of \( P_k \).

The value 2 in (1) is the smallest whole number we can get from the analysis below. Replacing it by a larger constant relaxes the constraints on \( d \).

## 2 Proof of Theorem 1

We will w.l.o.g. only prove (1) for the case \( k = n \). This is valid since we will see that our claims hold a fortiori for smaller \( k \).

We first observe that if \( R_{k-1} = \{y_1, y_2, \ldots, y_{k-1}\} \) then

\[ y_k \text{ is chosen uniformly from } \bar{R}_{k-1} \]

and is independent of the graph \( \Gamma_{k-1} \) induced by \( L_{k-1} \cup R_{k-1} \). This is because we can expose \( \Gamma \) along with the algorithm. When we start the construction of \( M_k \) we expose the neighbors of \( v_k \) one by one. In this way we either determine that \( S_k(v_k) = \emptyset \) or we expose a random member of \( S_k(v_k) \) without revealing anymore of \( N(v_k) \). In general, in Step 2, we have either exposed all the neighbors of \( x \) and these will necessarily be in \( R_{k-1} \). Or, we can proceed to expose the unexposed neighbors of \( x \) until either (i) we determine that \( S_k(x) = \emptyset \) and we choose a random member of \( N(x) \) or (ii) we find a neighbor of \( x \) that is a random member of \( \bar{R}_{k-1} \). Thus \( R_{n-1} \) is a random subset of \( R \).

Let

\[ B = \{v \in L : N(v) \cap \bar{R}_{n-1} = \emptyset\}. \]
If \( x \notin B \) in Step 2 of \textsc{insert} then we will have found \( P_n \).

Let \( P = (x_1, y_1, x_2, y_2, \ldots, x_\ell) \) be a path in \( \Gamma \), where \( x_1, x_2, \ldots, x_\ell \in L \) and \( y_1, y_2, \ldots, y_\ell-1 \in R \). We say that \( P \) is interesting if \( x_1, x_2, \ldots, x_\ell \in B \). We note that if the path \( P_n = (x_0 = v_n, y_0, x_1, y_1, \ldots, x_\ell, y_\ell, x_{\ell+1}, y_{\ell+1}) \) then \( Q_n = (x_1, y_1, x_2, y_2, \ldots, x_\ell) \) is interesting. Indeed, we must have \( x_i \in B, \quad 1 \leq i \leq \ell \), else \textsc{insert} would have chosen \( y_i \in \overline{R}_{n-1} \subseteq \overline{R}_{x_i} \) and completed the round.

Let \( \nu_\ell \) denote the number of interesting paths with \( 2\ell - 1 \) vertices.

**Lemma 2** There exists an absolute constant \( c_0 > 0 \) such that given \( A_0 \) and \( d \) sufficiently large,

\[
\Pr(\exists 2 \leq \ell \leq A_0 \log \log n : \nu_\ell \geq 12n e^{-\ell d / 3}) = o(n^{-2}).
\]

Here our probability is over \( \Gamma \) and the choice of \( R_{n-1} \).

Before proving the lemma, we show how it can be used to prove Theorem 1. We will need the following claim:

**Claim 3** W.h.p. \( \Gamma \) contains at most \( n^{1/2} \) cycles of length at most \( \lambda_d = (\log_d n)^{1/100} \).

**Proof of Claim:** The expected number of cycles of length at most \( 2\ell = \lambda_d \) is bounded by

\[
\binom{n}{\ell} \binom{m}{\ell} (\ell!)^2 \left( \frac{d}{m} \right)^{2\ell} \leq (de)^{2\ell} \leq n^{1/50}.
\]

The claim follows from the Markov inequality.

**End of proof of Claim**

**Claim 4** W.h.p. \( \Gamma \) has maximum degree at most \( \log n \).

**Proof of Claim:** If \( v \in L \) then its degree \( d(v) = d \). Now consider \( w \in R \). Then for \( \ell = \log n \),

\[
\Pr(\exists w \in R : d(w) \geq \ell) \leq m \binom{dn}{\ell} \frac{1}{m^\ell} \leq m \left( \frac{de}{\ell} \right)^{\ell} \leq n^{-\log \log n / 2}. \tag{3}
\]

**End of proof of Claim**

As a corollary of Claim 3 and Claim 4 we see that if \( 2\ell = (\log n)^{1/100} \),

w.h.p. there are at most \( n^{1/2} (\log n)^{2\ell} = n^{1/2 + o(1)} \) vertices within distance \( 2\ell \) of a cycle of length at most \( 2\ell \). \( \tag{4} \)

Continuing with the proof of Theorem 1 we will need the following result from [5]: We phrase Claim 10 of that paper in our current terminology.

**Claim 5** There exists a constant \( a > 0 \) such that for any \( v \in L_{n-1} \), the expected time for \textsc{insert} to reach \( \overline{R}_{n-1} \) is \( O((\log n)^a) \).
Now let \( p_\ell \) denote the probability that \textsc{insert} requires at least \( \ell \) rounds to insert \( v_{n-1} \). We prove the theorem by showing that
\[
E(|P_n|) = 1 + 2 \sum_{\ell=2}^{\infty} p_\ell \leq 2. \tag{5}
\]
(We only need to verify the second equation.)

We observe that if \( v_{n-1} \) has no neighbor in \( \bar{R}_{n-1} \) and has no neighbor in a cycle of length at most \( \lambda_d \) then for some \( \ell \), the first \( \ell \leq A_0 \log \log n \) vertices of \( P_n \) follow an interesting path. Hence,
\[
\sum_{\ell=2}^{L \log \log n} p_\ell \leq O(n^{-1/2+o(1)}) + \sum_{\ell=2}^{A_0 \log \log n} \frac{\nu_\ell}{n(d-1)^\ell} \leq O(n^{-1/2+o(1)}) + \frac{ne^{-c_0 \ell d/2}}{n(d-1)^\ell} \leq \frac{1}{3}. \tag{6}
\]

**Explanation of (6):** Following (4), we find that the probability \( v_n \) is within \( 2A_0 \log \log n \) of a cycle of length at most \( \lambda_d \) is bounded by \( n^{-1/2+o(1)} \). The \( O(n^{-1/2+o(1)}) \) term accounts for \( v_n \) choosing a vertex close to a short cycle. Failing this, we have divided the number of interesting paths of length \( 2\ell \) by the number of equally likely walks \( n(d-1)^\ell \) that \textsc{insert} could take.

Note next that
\[
P_{A_0 \log \log n} \leq O(n^{-1/2+o(1)}) + \frac{e^{-c_0 \ell d/2}}{(d-1)^{A_0 \log \log n}} \leq \frac{1}{(\log n)^{A_1}}
\]
where the constant \( A_1 \) can be made as large as necessary by increasing \( d \).

It follows that
\[
\sum_{\ell=A_0 \log \log n}^{A_0 \log \log n} p_\ell \leq \sum_{\ell=A_0 \log \log n}^{(\log n)^{A_1}} p_{A_0 \log \log n} \leq \frac{1}{3}. \tag{7}
\]

It follows from Claim 5 that for any integer \( \rho \geq 1 \),
\[
P(|P_n| \geq \rho(\log n)^{2a}) \leq \frac{1}{(\log n)^{\rho a}}. \tag{8}
\]

It follows from (8) that
\[
\sum_{\ell=1}^{(\log n)^{2a}} p_\ell \leq 1 + \sum_{\rho=1}^{\infty} \sum_{\ell/((\log n)^{2a} \in [\rho, \rho+1])} \frac{1}{(\log n)^{\rho a-2}} = o(1). \tag{9}
\]

Theorem 1 now follows from (5), (6), (7) and (9), if we take \( A_1 \geq 2a \).

## 2.1 Proof of Lemma 2

Fix \( 2 \leq \ell \leq A_0 \log \log n \) and let \( P_\ell \) denote the set of interesting paths of length \( 2\ell - 1 \). For \( P \in P_\ell \) we let \( d(P) \) denote the number of \( Q \in P_\ell \) such that \( P, Q \) share a vertex. Then we let \( \Delta_\ell = \max_{P \in P_\ell} d(P) \). The Markov inequality implies that that for any \( \alpha > 0 \) and \( t = 1, 2, \ldots, \)
\[
P(\nu_\ell \geq \alpha n) = P((\nu_\ell)_t \geq (\alpha n)_t) \leq \frac{E((\nu_\ell)_t)}{(\alpha n)_t}. \tag{10}
\]
Here \( (m)_t = m(m-1)\cdots(m-t+1) \).
We will use the inequality
\[ E(\nu_t) \leq \sum_{i=0}^{t-1} (\Delta(t-i))^i E(\nu_t)^{t-i}. \] (11)

The expectation on the LHS of (11) is of the number of distinct \( t \)-sequences of paths in \( P_\ell \). A summand bounds the expected number of sequences obtained by choosing \( t-i \) vertex disjoint paths and then choosing \( i \) paths that meet one of these \( t-i \) paths. The lemma will follow from
\[ P(\Delta \geq (\log n)^{2\ell}) \leq n^{-\omega} \text{ where } \omega \to \infty. \] (12)

Indeed, substituting (12), (13) into (11) gives that for \( t = (\log n)^{2\ell} \),
\[ E(\nu_t) \leq \beta n \text{ where } \beta = e^{-\varepsilon d/3}. \] (13)

Substituting (14) into (10) gives, for \( t = (\log n)^{2\ell} \) and \( \alpha = 12\beta \),
\[ P(\nu_t \geq \alpha n) \leq \frac{(\log n)^{2\ell}(\beta n)^{t-i}}{(\alpha n)^{t}} \leq \left( \frac{2e\beta}{\alpha} \right)^t \leq 2^{-t}. \]

Proof of (12)

It follows from (3) that with the claimed probability,
\[ \Delta \leq 2\ell d^\ell (\log n)^\ell \leq (\log n)^{2\ell}. \]

Proof of (13)

Claim 6 Let
\[ B = \left\{ |B| \geq ne^{-\varepsilon d/2} \right\}. \]

Then
\[ P(B) = O(n^{-2}). \]

Proof of Claim: First of all let
\[ K = \{ k : \text{ round } k \text{ does not end immediately in Step 2 with } x = v_k. \} \]

Then, \( P(k \in K) \leq \left( \frac{k}{m} \right)^d \) and this holds for each value of \( k \) independently and so
\[ E(|K|) \leq \sum_{k=1}^{n} \left( \frac{k}{m} \right)^d \leq \frac{n^{d+1}}{dm^d} \leq nd^{-1} e^{-\varepsilon d}. \]

Now \( |K| \) is the sum of independent \( \{0,1\} \) random variables and so Hoeffding’s theorem implies that
\[ P(|K| \geq 2nd^{-1} e^{-\varepsilon d}) = O(n^{-2}), \]
with room to spare.

Now if $B_1 = \{v_k \in B : \exists \ell \neq k \text{ s.t. round } \ell \text{ ends with } x = v_k \}$ then $|B_1| \leq |K|$. But if $B_2 = B \setminus B_1$ then $v_k \in B_2$ only if random choices of other vertices of $L$ include all the neighbors of $v_k$. Then, it follows from (2) that

$$P(v_k \in B_2) \leq \left(1 - \left(1 - \frac{1}{m}\right)^n\right)^d \leq (1 - \epsilon)^d.$$  

It is straightforward to show concentration of $|B_2|$ using the Azuma-Hoeffding inequality.

**End of proof of Claim**

Given Claim 6, we have

$$E(\nu_\ell) = E(\nu_\ell \mid \neg B)P(\neg B) + E(\nu_\ell \mid B)P(B)$$

$$\leq n^\ell e^{-\epsilon d\ell/2} m^{\ell-1} \cdot \left(\frac{(1 + o(1))d}{n-1}\right)^{2\ell-2} + O(n\ell e^{-\epsilon d/2})$$

$$\leq n((1 + o(1))d^2(1 - \epsilon)^{-1} e^{-\epsilon d/2})\ell + o(1),$$

$$\leq ne^{-\epsilon \ell d/3},$$

for $d$ sufficiently large.

**Explanation of (15):** We choose the vertex sequence $\sigma = (x_1, y_1, \ldots, y_{\ell-1}, x_\ell)$ of an interesting path $P$ in at most $|B|^\ell m^{\ell-1}$ ways. Having chosen $\sigma$ we see that $(1 + o(1))d/(n - 1))^{2\ell-2}$ bounds the probability that the edges of $P$ exist. To see this, condition on the random choices for vertices not on $P$. Let $M$ be the property that $\Gamma$ has a matching from $L$ to $R$. It is known that $P(M) = 1 - O(n^{-d})$. So for large $d$, we can assume that our conditioning is such that almost all edge choices by $x_1, x_2, \ldots, x_\ell$ are such that $\Gamma$ has property $M$. Recall from (2) that the disposition of the edges of $\Gamma_{n-1}$ is independent of $R_{n-1}$. Now, each edge adjacent to a given $x \in P \cap L$ is a uniform choice over those edges consistent with $x$ being in $B$. But there will always be at least $n - 1$ such choices for such an $x$. Thus

$$P(x \text{ chooses } y \mid M) \leq \frac{P(x \text{ chooses } y)}{P(M)} \leq (1 + o(1)) \frac{d}{n-1}.$$  

This argument applies equally well to the $2\ell - 2$ simultaneous choices of $x_1, x_2, \ldots, x_\ell$ and the bound follows.

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**References**


