The cover time of random regular graphs

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Abstract

Let $r \geq 3$ be constant, and let $\mathcal{G}_r$ denote the set of r-regular graphs with vertex set $V = \{1, 2, \ldots, n\}$. Let G be chosen randomly from $\mathcal{G}_r$. We prove that **whp** the cover time of a random walk on G is asymptotic to $\frac{r-1}{r-2}\, n \log n$.

1 Introduction

Let $G = (V, E)$ be a connected graph, let $|V| = n$, and $|E| = m$. A *random walk* $\mathcal{W}_u$, $u \in V$ on the undirected graph $G = (V, E)$ is a Markov chain $X_0 = u, X_1, \ldots, X_t, \ldots \in V$ associated to a particle that moves from vertex to vertex according to the following rule: the probability of a transition from vertex i, of degree d_i, to vertex j is $1/d_i$ if $\{i, j\} \in E$, and 0 otherwise. For $u \in V$ let C_u be the expected time taken for $\mathcal{W}_u$ to visit every vertex of G. The *cover time* C_G of G is defined as $C_G = \max_{u \in V} C_u$. The cover time of connected graphs has been extensively studied. It is a classic result of Aleliunas, Karp, Lipton, Lovász and Rackoff [2] that $C_G \leq 2m(n-1)$. It was shown by Feige [8], [9], that for any connected graph G

$$(1 - o(1))n \log n \leq C_G \leq (1 + o(1))\frac{4}{27}n^3.$$

The lower bound is achieved by (for example) the complete graph K_n, whose cover time is determined by the Coupon Collector problem.

In a previous paper [7] we studied the cover time of random graphs $G_{n,p}$ when $np = c \log n$ where $c = O(1)$ and $(c-1)\log n \to \infty$. This extended a result of Jonasson, who proved in [12] that when the expected average degree $(n-1)p$ grows faster than $\log n$, **whp** a random

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graph has the same cover time (asymptotically) as the complete graph K_n, whereas, when $np = \Omega(\log n)$ this is not the case.
(A sequence of events $\mathcal{E}_n, n \geq 0$, is said to occur *with high probability* (**whp**), if $\lim_{n\to\infty} \mathbf{Pr}(\mathcal{E}_n) = 1$.)

Theorem 1. *[7] Suppose that $np = c \log n = \log n + \omega$ where $\omega = (c-1)\log n \to \infty$ and $c \geq 1$. If $G \in G_{n,p}$, then* **whp**

$$C_G \sim c \log\left(\frac{c}{c-1}\right) n \log n.$$

The notation $A_n \sim B_n$ means that $\lim_{n\to\infty} A_n/B_n = 1$.

The main new result of the paper concerns the cover time of random regular graphs.

Theorem 2. *Let $r \geq 3$ be constant. Let $\mathcal{G}_r$ denote the set of r-regular graphs with vertex set $V = \{1, 2, \ldots, n\}$. If G is chosen randomly from $\mathcal{G}_r$, then* **whp**

$$C_G \sim \frac{r-1}{r-2} n \log n.$$

Using a similar argument we can consider how many steps are needed for the walk to get within distance k of every vertex. Let us call this $C_G^{(k)}$. Cover time corresponds to $k = 0$. We prove

Theorem 3. *Let $r \geq 3$, $k \geq 0$ be constants. Let $\mathcal{G}_r$ denote the set of r-regular graphs with vertex set $V = \{1, 2, \ldots, n\}$. If G is chosen randomly from $\mathcal{G}_r$, then* **whp**

$$C_G^{(k)} \sim \frac{1}{(r-2)(r-1)^{k-1}} n \log n.$$

The next section contains the heart of the proof of our Theorems. In it we establish a good estimate of the probability that the first visit of $\mathcal{W}$ to a vertex v takes place at a time t. Once this is done, we can proceed to the proof of Theorem 2 in Section 3 and the proof of Theorem 3 in Section 6.

2 The first visit time lemma.

2.1 Convergence of the random walk

In this section G denotes a fixed connected graph, and u is some arbitrary vertex from which a walk $\mathcal{W}_u$ is started. Let $\mathcal{W}_u(t)$ be the vertex reached at step t, let P be the matrix of transition probabilities of the walk and let $P_u^{(t)}(v) = \mathbf{Pr}(\mathcal{W}_u(t) = v)$. Let $\pi_v = \frac{d_v}{2m}$ for $v \in V$.

Let $\lambda_{\max} > 0$ be the second largest absolute value of an eigenvalue of P. Assume that $\lambda_{\max} < 1$. Then,

$$|P_u^{(t)}(x) - \pi_x| \leq (\pi_x/\pi_u)^{1/2}\lambda_{\max}^t \leq n^{1/2}\lambda_{\max}^t. \tag{1}$$

See for example [11]. (Note that connectivity and $\lambda_{\max} < 1$ implies ergodocity).

2.2 Generating function formulation

For the results of this section, we do not require that G be regular.

Fix two vertices u, v. Let h_t be the probability $\mathbf{Pr}(\mathcal{W}_u(t) = v) = P_u^{(t)}(v)$, that the walk $\mathcal{W}_u$ visits v at step t. Let $H(s)$ be the generating function for the sequence $h_t, t \geq 0$.

Similarly, considering the walk $\mathcal{W}_v$, starting at v, let r_t be the probability that this walk returns to v at step $t = 0, 1, ...$ Let $R(s)$ be the generating function for the sequence $r_t, t \geq 0$. We note that $r_0 = 1$.

Let $f_t(u{\to}v)$ be the probability that the first visit of the walk $\mathcal{W}_u$ to v occurs at step t. If $u \neq v$ then $f_0(u{\to}v) = 0$. Let $F(s)$ generate $f_t(u{\to}v)$. Thus

$$H(s) = F(s)R(s). \tag{2}$$

Let

$$T = \frac{4\log n}{\log 1/\lambda_{\max}}. \tag{3}$$

We note that (1) gives

$$\max_{x\in V} |P_u^{(t)}(x) - \pi_x| \leq n^{-3} \qquad \text{for } t \geq T. \tag{4}$$

For $R(s)$ let

$$R_T(s) = \sum_{j=0}^{T-1} r_j s^j. \tag{5}$$

Thus $R_T(s)$ generates the probability of a return to v during steps $0, ..., T-1$ of a walk starting at v. Similarly for $H(s)$, let

$$H_T(s) = \sum_{j=0}^{T-1} h_j s^j. \tag{6}$$

2.3 First visit time: Single vertex v

The following lemma should be viewed in the context that G is an n vertex graph which is part of a sequence of graphs with n growing to infinity. We prove it in greater generality than is needed for the proof of Theorem 2.

In what follows $c_1, c_2, \ldots$ are positive constants independent of n.

Lemma 4. *Let T be as defined in (3). Suppose that*

(a) $H_T(1) \leq (1-c_1)R_T(1)$.

(b) $\max\limits_{|s|=1} \frac{|R_T(s)-R_T(1)|}{R_T(1)} \leq 1-c_2$.

(c) $T\pi_v = o(1)\,,\; T\pi_v = \Omega(n^{-2})$.

(d) $\lambda_{\max} \leq c_3 < 1$.

Let

$$\lambda = \frac{c_2}{100T}. \tag{7}$$

Let

$$p_v = \frac{\pi_v}{R_T(1)(1+O(T\pi_v))}, \tag{8}$$

$$c_{u,v} = 1 - \frac{H_T(1)}{R_T(1)(1+O(T\pi_v))}, \tag{9}$$

where the values of the $1+O(T\pi_v)$ terms are given implicitly in (16), (19) respectively. Then

$$f_t(u{\to}v) = c_{u,v}\frac{p_v}{(1+p_v)^{t+1}} + O(e^{-\lambda t/2}) \qquad \text{for all } t \geq T. \tag{10}$$

Proof Write

$$R(s) = R_T(s) + \widehat{R}_T(s) + \frac{\pi_v s^T}{1-s}, \tag{11}$$

$$A(s) = (1-s)R(s) = \pi_v s^T + (1-s)(R_T(s) + \widehat{R}_T(s)), \tag{12}$$

where $R_T(s)$ is given by (5) and

$$\widehat{R}_T(s) = \sum_{t\geq T}(r_t - \pi_v)s^t$$

generates the error in using the stationary distribution π_v for r_t when $t \geq T$.

Note that while (11) is only valid for $|s| < 1$, the fact that $|r_t - \pi_v| \leq n^{1/2}c_3^t$ means that the expansion (12) is valid for $|s| < c_3^{-1}$.

Similarly, let

$$H(s) = H_T(s) + \widehat{H}_T(s) + \pi_v\frac{s^T}{1-s}, \tag{13}$$

$$B(s) = (1-s)H(s) = \pi_v s^T + (1-s)(H_T(s) + \widehat{H}_T(s)). \tag{14}$$

Using (11), (13) we rewrite $F(s) = H(s)/R(s)$ from (2) as $F(s) = B(s)/A(s)$

If $|s| \le \lambda_{\max}^{-1/3}$ then (1) implies that for $Z = H, R$,

$$|\widehat{Z}(s)| \le n^{1/2} \sum_{t \ge T} (s\lambda_{\max})^t = o(n^{-2}). \tag{15}$$

For real $s \ge 1$ and $Z = H, R$, we have

$$Z_T(1) \le Z_T(s) \le Z_T(1)s^T.$$

Let $s = 1 + \beta\pi_v$, where $\beta > 0$ is constant. Since $T\pi_v = o(1)$ we have

$$Z_T(s) = Z_T(1)(1 + O(T\pi_v)).$$

$T\pi_v = o(1)$ implies that $|s| \le \lambda_{\max}^{-1/3}$ and (15) applies. As $T\pi_v = \Omega(n^{-2})$ and $R_T(1) \ge 1 + r_2 > 1 + \frac{1}{n}$ this implies that

$$A(s) = \pi_v(1 - \beta R_T(1)(1 + O(T\pi_v))).$$

It follows that $A(s)$ has a real zero at s_0, where

$$s_0 = 1 + \frac{\pi_v}{R_T(1)(1 + O(T\pi_v))} = 1 + p_v, \tag{16}$$

say. We also see that

$$A'(s_0) = -R_T(1)(1 + O(T\pi_v)) \ne 0 \tag{17}$$

and thus s_0 is a simple zero (see e.g. [6] p193). The value of $B(s)$ at s_0 is

$$B(s_0) = \pi_v \left(1 - \tfrac{H_T(1)}{R_T(1)(1+O(T\pi_v))} + O(T\pi_v)\right) \ne 0. \tag{18}$$

Thus, from (8), (9)

$$\frac{B(s_0)}{A'(s_0)} = -p_v c_{u,v}. \tag{19}$$

Thus the residue of $F(s)$ at s_0 is $B(s_0)/A'(s_0)$ (see e.g. [6] p195) and the principal part of the Laurent expansion of $F(s)$ at s_0 is

$$f(s) = \frac{B(s_0)/A'(s_0)}{s - s_0}. \tag{20}$$

To approximate the coefficients of the generating function $F(s)$, we now use a standard technique for the asymptotic expansion of power series (see e.g.[13] Th 5.2.1).

We prove (below) that s_0 is the only zero of $A(s)$ inside the circle $C_\lambda = \{s = (1 + \lambda)e^{i\theta}\}$. Thus $F(s) = f(s) + g(s)$, where $g(s)$ is analytic in C_λ. Let $M = \max_{s \in C_\lambda} |g(s)|$. Thus $M \le \max |f(s)| + \max |F(s)|$.

As $F(s) = B(s)/A(s)$ on C_λ we have that

$$|F(s)| \leq \frac{H_T(1)(1+\lambda)^T + O(T\pi_v)}{|R_T(s)| - O(T\pi_v)}.$$

Let $\widetilde{s} = s/(1+\lambda)$. We note that $|R_T(s) - R_T(\widetilde{s})| \leq ((1+\lambda)^T - 1)R_T(1)$ and also that (b) implies that $|R_T(\widetilde{s})| \geq c_2 R_T(1)$ which implies that

$$|R_T(s)| \geq R_T(1)\left(c_2 + 1 - e^{c_2/100}\right),$$

and hence $M = O(1)$.

Let $a_t = [s^t]g(s)$, then (see e.g.[6] p143), $a_t = g^{(t)}(0)/t!$. By the Cauchy Inequality (see e.g. [6] p130) we have that $|g^{(t)}(0)| \leq Mt!/(1+\lambda)^t$ and thus

$$|a_t| \leq \frac{M}{(1+\lambda)^t} = O(e^{-t\lambda/2}).$$

As $[s^t]F(s) = [s^t]f(s) + [s^t]g(s)$ and $[s^t]1/(s - s_0) = -1/(s_0)^{t+1}$ we have

$$[s^t]F(s) = \frac{-B(s_0)/A'(s_0)}{s_0^{t+1}} + O(e^{-t\lambda/2}). \tag{21}$$

Thus, we obtain

$$[s^t]F(s) = c_{u,v}\frac{p_v}{(1+p_v)^{t+1}} + O(e^{-t\lambda/2}),$$

which completes the proof of (10).

We now prove that s_0 is the only zero of $A(s)$ inside the circle C_λ. We use Rouché's Theorem (see e.g. [6]), the statement of which is as follows: *Let two functions $f(z)$ and $g(z)$ be analytic inside and on a simple closed contour C. Suppose that $|f(z)| > |g(z)|$ at each point of C, then $f(z)$ and $f(z) + g(z)$ have the same number of zeroes, counting multiplicities, inside C.*

Let the functions $f(s), g(s)$ be given by $f(s) = (1-s)R_T(1)$ and $g(s) = \pi_v s^T + (1-s)(R_T(s) - R_T(1) + \widehat{R}_T(s))$. For $s \in C_\lambda$, let $\widetilde{s} = s/(1+\lambda)$,

$$\begin{aligned} |g(s)|/|f(s)| &\leq \frac{\pi_v(1+\lambda)^T}{\lambda R_T(1)} + \frac{|R_T(s) - R_T(\widetilde{s})|}{R_T(1)} + \frac{|R_T(\widetilde{s}) - R_T(1)|}{R_T(1)} + o(n^{-2}) \\ &\leq 100e^{c_2/100}\pi_v T + \left(e^{c_2/100} - 1\right) + (1 - c_2) + o(n^{-2}) \\ &< 1. \end{aligned}$$

As $f(s) + g(s) = A(s)$ we conclude that $A(s)$ has only one zero inside the circle C_λ. This is the simple zero at s_0. □

Corollary 5. *Let $\boldsymbol{A}_t(v)$ be the event that $\mathcal{W}_u$ has not visited v by step t. Then for $t \geq T$,*

$$\mathbf{Pr}(\boldsymbol{A}_t(v)) = \frac{c_{u,v}}{(1+p_v)^t} + O(\lambda^{-1}e^{-\lambda t/2}).$$

Proof We use Lemma 4 and $\mathbf{Pr}(\boldsymbol{A}_t(v)) = \sum_{\tau > t} f_\tau(u \to v)$. □

3 Random regular graphs are nice

Our task now is to show that a typical r-regular graph satisfies the conditions (a) – (d) of Lemma 4 and to compute $R_T(1)$.

We start with some typical properties of a random regular graph. Let

$$\sigma = \lfloor \log\log\log n \rfloor.$$

Say a cycle C is *small* if $|C| \leq \sigma$.

An r-regular graph G is *nice* if

P1. G is connected.

P2. The second eigenvalue of the adjacency matrix of G is at most $2\sqrt{r-1}+\epsilon$, where $\epsilon > 0$ is arbitrarily small ($\epsilon = 1/10$ is small enough).

P3. There are at most $r^{2\sigma}$ vertices on small cycles.

P4. No pair of small cycles are within distance 3σ of each other.

Theorem 6. *Let $r \geq 3$ be a constant and let G be chosen uniformly from the set $\mathcal{G}_r$ of r-regular graphs with vertex set $[n]$. Then G is nice* **whp**.

Proof
(**P1**) That a random r-regular graph is r-connected **whp**, for $r \geq 3$, was proved in [4].

(**P2**) That the second eigenvalue of a random r-regualr graph is this small **whp** was proved by Friedman [10].

For **P3,P4** we use the configuration model as elaborated in [5]. Let $W = [n] \times [r]$ ($W_v = v \times [r]$ represents r *half edges* incident with vertex $v \in [n]$.) A *configuration* F is a partition of W into $rn/2$ 2-element subsets and Ω denotes the set of possible configurations. We associate with F a multigraph $\mu(F) = ([n], E(F))$ where, as a multi-set,

$$E(F) = \{(v,w) : \{(v,i),(w,j)\} \in F \text{ for some } 1 \leq i,j \leq r\}.$$

(Note that $v = w$ is possible here.)

We say that F is *simple* if the multigraph $\mu(F)$ has no loops or multiple edges. Let Ω_0 denote the set of simple configurations. It is known that if F is chosen uniformly from Ω then

(a) Each $G \in \mathcal{G}(n,r)$ is the image (under μ) of exactly $(r!)^n$ simple configurations.

(b) $\mathbf{Pr}(F \in \Omega_0) \approx e^{-(r^2-1)/4}$.

It follows from this that any property of almost every $\mu(F)$ is a property of almost every member of $\mathcal{G}_r$.

(**P3**) The expected number of small cycles in $\mu(F)$, F chosen randomly from Ω, is bounded by

$$\sum_{k=3}^{\sigma} \binom{n}{k} \frac{(k-1)!}{2} \frac{(r(r-1))^k M_{rn-2k}}{M_{rn}} \leq \sum_{k=3}^{\sigma} \binom{n}{k} \frac{(k-1)!}{2} \left(\frac{r}{n}\right)^k \leq r^{\sigma},$$

where $M_{2m} = \frac{(2m)!}{2^m m!}$ and $|\Omega| = M_{rn}$.

The almost sure occurrence of property **P3** now follows from the Markov inequality.

(**P4**) Similarly, the expected number of pairs of small cycles which are close to each other is bounded by

$$\sum_{a=3}^{\sigma}\sum_{b=3}^{\sigma} \binom{n}{a}\binom{n}{b-1} \frac{(a-1)!}{2} \frac{(b-1)!}{2} \left(\frac{r}{n}\right)^{a+b} +$$
$$\sum_{a=3}^{\sigma}\sum_{b=3}^{\sigma}\sum_{c=1}^{\sigma} \binom{n}{a}\binom{n}{b}\binom{n}{c} \frac{(a-1)!}{2} \frac{(b-1)!}{2} ab \left(\frac{r}{n}\right)^{a+b+c+1} = o(1).$$

□

Remark 1. *Although the main subject of the paper is random regular graphs, it is worth mentioning* Ramanujan graphs. *An n-vertex r-regular graph is Ramanujan if $\lambda_{\max} \leq \frac{2\sqrt{r-1}}{r}$. It is known that such graphs have girth $\Omega(\log n)$ and so they are nice, see Alon [3]. Consequently, their cover time $\sim \frac{r-1}{r-2} n \log n$.*

Remark 2. *Aldous [1] considered the cover time of Cayley graphs and obtained a similar expression for the cover time. By relaxing the assumptions in Lemma 4 it is possible to obtain some of his results e.g. the hypercube and toroidal grids in three or more dimensions.*

4 Nice graphs

Assume from now on that G is a nice regular graph. For $v \in V$ and $k \geq 0$, let $N_k(v) = \{w : dist(v,w) = k\}$ be the set of vertices at distance k from v. Let $M_l(v) = \cup_{j=0}^{l} N_l(v)$, and let $G_l(v)$ be the subgraph of G induced by $M_l(v)$. Also let us replace the notations $R_T(1), H_T(1)$ by R_v, H_v reflecting their dependence on v.

Definition 1. *We say v is locally tree-like if $G_\sigma(v)$ is a tree.*

Lemma 7. *If v is locally tree-like then*

$$R_T(1) = \frac{r-1}{r-2} + o(\sigma^{-1}).$$

Proof Let T_r be the infinite r-regular tree, rooted at v. Let $\mathcal{X}$ be a random walk on T_r starting at v. Let ρ_i be the probability that $\mathcal{X}$ is at v at step i. Now we can project the walk $\mathcal{X}$ onto a walk $\mathcal{Y}$ on $\{0, 1, 2, \ldots, \}$ where the particle moves right with probability $q = \frac{r-1}{r}$ and left with probability $p = \frac{1}{r}$, except of course at the origin, where it must move right. Let E_i be the expected number of visits to 0 for $\mathcal{Y}$ starting at i. Then

$$E_0 = 1 + E_1 = 1 + E_0 p/q.$$

This is because E_1 is E_0 times the expected number of visits to 0 between right moves from 1. Solving gives

$$\sum_{i=0}^{\infty} \rho_i = E_0 = \frac{r-1}{r-2}. \tag{22}$$

Note next that for $i \geq 0$ we have $\rho_{2i+1} = 0$ and we will argue that

$$\rho_{2i} \leq \binom{2i}{i} \frac{(r-1)^i}{r^{2i}}. \tag{23}$$

and then

$$\sum_{i=\sigma+1}^{\infty} \rho_i \leq \sum_{j=\sigma/2} \binom{2j}{j} \frac{(r-1)^i}{r^{2i}} = o(\sigma^{-1}). \tag{24}$$

We compare this with $R_T(1)$. First observe that $r_i = \rho_i$ for $i \leq \sigma$. Then from (1) we see that

$$\sum_{i=\sigma+1}^{T} r_i \leq \sum_{i=\sigma+1}^{T} (\pi_v + \lambda_{\max}^i) = o(\sigma^{-1}).$$

Let us now prove (23). First observe that the RHS of (23) is the probability that a walk $\mathcal{Y}_1$ is at the origin after $2i$ steps. Here $\mathcal{Y}_1$ is the walk on $\{0, \pm 1, \pm 2, \ldots, \}$ where the particle moves right with probability $q = \frac{r-1}{r}$ and left with probability $p = \frac{1}{r}$ i.e. there is no barrier at the origin. We can couple $\mathcal{Y}, \mathcal{Y}_1$ so that $\mathcal{Y}(t) \geq |\mathcal{Y}_1(t)|$. When $\mathcal{Y}_1(t) > 0$ we can move them in the same direction and when $\mathcal{Y}_1 < 0$ then we can move $\mathcal{Y}$ further from the origin whenever $\mathcal{Y}_1$ moves further from the origin.

The lemma now follows from (22) and (24). □

Remark 1. *Because there are very few non-tree-like vertices and because they are far apart, we will find that we do not need to estimate $R_T(1)$ for such vertices. It is relatively easy to show that for non-tree-like vertices $R_T(1) = 1 + O(r^{-1})$ as $r \to \infty$, thus the only difficulty is with small r.*

Lemma 8. *If v is locally tree-like then for $|s| = 1$, $\frac{|R_T(s) - R_T(1)|}{R_T(1)} \leq \frac{5}{6}$.*

Proof For any s,

$$|R_T(s) - R_T(1)| \leq \sum_{j=1}^{T} r_j |s^j - 1|.$$

As $|s| = 1$ we have that

$$\sum_{j=1}^{T} r_j |s^j - 1| \leq 2 \sum_{j=1}^{T} r_j. \tag{25}$$

We prove the lemma for $r \geq 4$ by observing that Lemma 7 implies

$$2\sum_{j=1}^{T} r_j = 2(R_T(1) - 1) = (1 + o(1))\frac{2}{r-2} \leq (1 + o(1))\frac{2}{3} \cdot \frac{r-1}{r-2} = (1 + o(1))\frac{2}{3}R_T(1). \tag{26}$$

When $r = 3$ we improve on (25) using ad-hoc arguments. First observe that $\pi_v = 1/n$ for $v \in V$ and that (1) implies that

$$S_0 = \sum_{i=\sigma}^{T} r_j |s^j - 1| \leq 2\sum_{i=\sigma}^{T} r_i \leq 2\sum_{i=\sigma}^{T} (\lambda_{\max}^i + \pi_v) = o(1). \tag{27}$$

Now consider $j < \sigma$. For a locally tree-like vertex, $r_j = 0$ if j is odd, and $r_j > 0$ if j is even. Fix $0 \leq \theta < 2\pi$ and let $s = e^{i\theta}$, then for $j = 2k$

$$|s^j - 1| = (2(1 - \cos j\theta))^{1/2} = 2|\sin k\theta|.$$

Thus

$$S_1 = \sum_{j=1}^{\sigma-1} r_j |s^j - 1| = 2 \sum_{k=1}^{\lfloor(\sigma-1)/2\rfloor} r_{2k} |\sin k\theta|.$$

Note now that $r_2 = \frac{1}{3}$ and $r_4 = \frac{5}{27}$. Suppose first that $\theta \notin I = [\frac{3\pi}{16}, \frac{5\pi}{16}] \cup [\frac{11\pi}{16}, \frac{13\pi}{16}]$. Then $|\sin 2\theta| \leq \sin \frac{3\pi}{8}$ and so

$$S_1 \leq 2\sum_{j=1}^{\sigma-1} r_j - \frac{2}{3}\left(1 - \sin\frac{3\pi}{8}\right). \tag{28}$$

On the other hand, if $\theta \in I$ then $|\sin 4\theta| \leq \sin \frac{\pi}{4}$ and then

$$S_1 \leq 2\sum_{j=1}^{\sigma-1} r_j - \frac{10}{27}\left(1 - \sin\frac{\pi}{4}\right). \tag{29}$$

(27), (28), (29) imply that $S_0 + S_1 \leq 2(R_T(1) - 1) - 1/3$. The lemma follows, since $R_T(1) \sim 2$ for $r = 3$. □

Finally we note:

Lemma 9. *For nice graphs,* $\frac{H_T(1)}{R_T(1)} \leq \frac{9}{10}$.

Proof Let f_t' be the probability that $\mathcal{W}_u$ has a first visit to v at time t. As $H(s) = F(s)R(s)$ we have

$$\begin{aligned} H_T(1) &\leq \mathbf{Pr}(\mathcal{W}_u \text{ visits } v \text{ by time } T-1)R_T(1) \\ &= R_T(1)\sum_{t=1}^{T-1} f_t'. \end{aligned}$$

Now (1) implies that if $\tau_0 = \lfloor 2 \log \lambda_{\max}^{-1} \log\log n \rfloor$ then

$$\sum_{t=\tau_0}^{T-1} f_t' \leq \sum_{t=\tau_0}^{T-1} (\pi_v + \lambda_{\max}^t) = o(1).$$

We now estimate $\sum_{t=0}^{\tau_0} f_t'$, the probability that $\mathcal{W}_u$ visits v by time τ_0. Let $v_1, v_2, \ldots, v_r$ be the neighbours of v and let w be the first neighbour of v visited by $\mathcal{W}_u$. Then

$$\begin{aligned} \mathbf{Pr}(\mathcal{W}_u \text{ visits } v \text{ by time } \tau_0) &= \sum_{i=1}^{r} \mathbf{Pr}(\mathcal{W}_u \text{ visits } v \text{ by time } \tau_0 \mid w = v_i)\mathbf{Pr}(w = v_i) \\ &\leq \sum_{i=1}^{r} \mathbf{Pr}(\mathcal{W}_{v_i} \text{ visits } v \text{ by the time } \tau_0)\mathbf{Pr}(w = v_i). \end{aligned}$$

So it suffices to prove the lemma when u is a neighbour of v. If $G_l(u)$ is a tree then we can argue as in Lemma 7. Let ψ be the probability that a particle at the root of T_r ever returns to the root. The expected number of visits is

$$\frac{r-1}{r-2} = \sum_{k=1}^{\infty} k\psi^{k-1}(1-\psi) = \frac{1}{1-\psi}.$$

So $\psi = \frac{1}{r-1}$ and

$$\mathbf{Pr}(\mathcal{W}_u \text{ does not visit } v \text{ by time } \tau_0) \geq \frac{r-1}{r}(1-\psi-o(1)) = \frac{r-2}{r} - o(1).$$

If $G_l(u)$ contains a cycle C then let $e = (\xi, \eta)$ be an edge of C not incident with u and let T_u be the tree $G_l(u) - e$. Let $N'(u) = \{u_1, u_2, \ldots, u_s\}$, $s \in \{r-2, r-1\}$ be the neighbours of u which are not on a shortest path from ξ or η to u in T_u. $|N'(u) \setminus \{v\}| \geq r-3$ and so

$$\mathbf{Pr}(\mathcal{W}_u \text{ does not visit } v \text{ by time } \tau_0) \geq \frac{r-3}{r}(1-\psi-o(1)) = \frac{(r-2)(r-3)}{r(r-1)} - o(1).$$

This leaves the case $r = 3$ and $N'(u) = \{v\}$. With probability $\frac{2}{3}$ we have $\mathcal{W}_u(1) \neq v$. If ξ or η is reached (possibly $N(u) = \{v, \xi, \eta\}$), then with probability $\frac{1}{3}$ the next move is away from u and $1 - \psi - o(1)$ bounds the probability that there is no return to ξ or η. Hence

$$\mathbf{Pr}(\mathcal{W}_u \text{ does not visit } v \text{ by time } \tau_0)) \geq \frac{2}{9}(1-\psi-o(1))$$

completing the proof of the lemma. □

5 Cover time of nice graphs

We now prove that

$$C_G \sim \frac{r-1}{r-2} n \log n.$$

Assume that $u, v \in V$ and that v is tree-like. Section 3 establishes that the conditions of Lemma 4 hold, and gives values for the parameters c_{uv}, p_v given by (8), (9). To summarize we have

$$R_T(1) = \frac{r-1}{r-2} + o(1), \qquad \frac{H_T(1)}{R_T(1)} \le \frac{9}{10}, \qquad \lambda_{\max} \le \frac{2\sqrt{r-1}+.1}{r},$$
$$\pi_v = \frac{1}{n}, \qquad T = O(\log n) \qquad \lambda = \Omega(1/\log n).$$

Hence, the probability that $\mathcal{W}_u$ has not visited v by some step $t \ge T$ (see Corollary 5) is given by

$$\mathbf{Pr}(\boldsymbol{A}_t(v)) = (1+o(1)) c_{uv} e^{-tp_v} + O(\lambda^{-1} e^{-\lambda t/2}).$$

Here $c_{uv} < 1$ and

$$p_v = \frac{r-2}{(r-1)n}(1+o(\sigma^{-1})).$$

5.1 Upper bound on cover time

Let $t_0 = \lceil (1+\sigma^{-1}) \frac{r-1}{r-2} n \log n \rceil$. We prove that for nice graphs, for any vertex $u \in V$,

$$C_u \le t_0 + o(t_0). \tag{30}$$

Let $T_G(u)$ be the time taken to visit every vertex of G by the random walk $\mathcal{W}_u$. Let U_t be the number of vertices of G which have not been visited by $\mathcal{W}_u$ at step t. We note the following:

$$C_u = \mathbf{E}\, T_G(u) = \sum_{t>0} \mathbf{Pr}(T_G(u) \ge t), \tag{31}$$

$$\mathbf{Pr}(T_G(u) > t) = \mathbf{Pr}(U_t > 0) \le \min\{1, \mathbf{E}\, U_t\}. \tag{32}$$

It follows from (31), (32) that for all t

$$C_u \le t + \sum_{s \ge t} \mathbf{E}\, U_s = t + \sum_{v \in V} \sum_{s \ge t} \mathbf{Pr}(\boldsymbol{A}_s(v)). \tag{33}$$

Let V_1 be the set of locally tree-like vertices and let $V_2 = V - V_1$. If G is nice then $|V_2| \le r^{3\sigma}$ for there are at most r^σ vertices within distance σ of a particular vertex in a small cycle, and at most $r^{2\sigma}$ vertices on small cycles.

For $v \in V_1$ we have

$$\begin{aligned}
\sum_{s \geq t_0} \mathbf{Pr}(\boldsymbol{A}_s(v)) &\leq (1+o(1))e^{-t_0 p_v} \sum_{s \geq t_0} e^{-(s-t_0)p_v} + O(\lambda^{-2} e^{-\lambda t_0/2}) \\
&\leq 2p_v^{-1} e^{-t_0 p_v} \\
&\leq 3\frac{r-1}{r-2}.
\end{aligned}$$

Furthermore, we see that in particular,

$$\mathbf{Pr}(\boldsymbol{A}_{5n}(v)) \leq 2e^{-1}. \tag{34}$$

Suppose next that $v \in V_2$. We can find $w \in V_1$ such that $dist(v, w) \leq \sigma$. So from (34), with $\nu = 5n + \sigma$, we have

$$\mathbf{Pr}(\boldsymbol{A}_\nu(v)) \leq 1 - (1 - 2e^{-1})r^{-\sigma}$$

since if our walk visits w, it will with probability at least $r^{-\sigma}$ visit v within the next σ steps. Thus if $\gamma = (1 - 2e^{-1})r^{-\sigma}$,

$$\begin{aligned}
\sum_{s \geq t_0} \mathbf{Pr}(\boldsymbol{A}_s(v)) &\leq \sum_{s \geq t_0} (1-\gamma)^{\lfloor s/\nu \rfloor} & (35) \\
&\leq \sum_{s \geq t_0} (1-\gamma)^{s/(2\nu)} \\
&= \frac{(1-\gamma)^{t_0/(2\nu)}}{1-(1-\gamma)^{1/(2\nu)}} \\
&\leq 3\nu\gamma^{-1}. & (36)
\end{aligned}$$

Thus, for all $u \in V$,

$$\begin{aligned}
C_u &\leq t_0 + 3\frac{r-1}{r-2}|V_1| + 3|V_2|\nu\gamma^{-1} \\
&= t_0 + O(r^{4\sigma} n) \\
&= t_0 + o(t_0),
\end{aligned}$$

as $\sigma = \lfloor \log \log \log n \rfloor$.

5.2 Lower bound on cover time

For any vertex u, we can find a set of vertices S such that at time $t_1 = t_0(1-\epsilon)$, $\epsilon \to 0$, the probability the set S is covered by the walk $\mathcal{W}_u$ tends to zero. Hence $T_G(u) > t_1$ **whp** which implies that $C_G \geq t_0 - o(t_0)$.

We construct S as follows. Let $S \subseteq V_1$ be some maximal set of locally tree-like vertices all of which are at least distance $2\sigma + 1$ apart. Thus $|S| \geq (n - r^{3\sigma})r^{-(2\sigma+1)}$.

Let $S(t)$ denote the subset of S which has not been visited by $\mathcal{W}_u$ after step t. Now, provided $t \geq T$

$$\mathbf{E}\ |S(t)| \geq (1-o(1)) \sum_{v \in S} \left(\frac{c_{u,v}}{(1+p_v)^t} + o(n^{-2}) \right).$$

Let u be a fixed vertex of S. Let $v \in S$ and let $H_T(1)$ be given by (6), then (1) implies that

$$H_T(1) \leq \sum_{t=\sigma}^{T-1} (\pi_v + \lambda_{\max}^t) = o(1). \tag{37}$$

Thus $c_{uv} = 1 - o(1)$. Setting $t = t_1 = (1-\epsilon)t_0$ where $\epsilon = 2\sigma^{-1}$, we have

$$\begin{aligned} \mathbf{E}\ |S(t_1)| &= (1+o(1))|S|e^{-(1-\epsilon)t_0 p_v} \\ &\geq n^{1/\sigma}. \end{aligned} \tag{38}$$

Let $Y_{v,t}$ be the indicator for the event that $\mathcal{W}_u$ has not visited vertex v at time t. Let $Z = \{v, w\} \subset S$. We will show (below) that that for $v, w \in S$

$$\mathbf{E}\ (Y_{v,t_1} Y_{w,t_1}) = \frac{c_{u,Z}}{(1+p_Z)^{t+2}} + o(n^{-2}), \tag{39}$$

where $c_{u,Z} \sim 1$ and $p_Z \sim 2(r-2)/(n(r-1))$. Thus

$$\mathbf{E}\ (Y_{v,t_1} Y_{w,t_1}) = (1+o(1))\mathbf{E}\ (Y_{v,t_1})\mathbf{E}\ (Y_{w,t_1}). \tag{40}$$

It follows from (38) and (40), that

$$\mathbf{Pr}(S(t_1) \neq 0) \geq \frac{(\mathbf{E}\ |S(t_1)|)^2}{\mathbf{E}\ |S(t_1)|^2} = \frac{1}{\frac{\mathbf{E}|S_{t_1}|(|S_{t_1}|-1)}{(\mathbf{E}|S(t_1)|)^2} + (\mathbf{E}\ |S_{t_1}|)^{-1}} = 1 - o(1).$$

Proof of (39). Let Γ be obtained from G by merging v, w into a single node Z. This node has degree $2r$ and every other node has degree r.

There is a natural measure preserving mapping from the set of walks in G which start at u and do not visit v or w, to the corresponding set of walks in Γ which do not visit Z. Thus the probability that $\mathcal{W}_u$ does not visit v or w in the first t steps is equal to the probability that a random walk $\widehat{\mathcal{W}_u}$ in Γ which also starts at u does not visit Z in the first t steps.

We apply Lemma 4 to Γ. That $\pi_Z = \frac{2}{n}$ is clear, and $c_{u,Z} = 1 - o(1)$ is argued as in (37). The derivation of $R_T(1)$ in Lemma 7 is also valid. The vertex Z is tree-like up to distance σ in Γ. The fact that the root vertex of the corresponding infinite tree has degree $2r$ does not affect the calculation of $R_T(1)$. □

6 Looking ahead

We now consider Theorem 3. Fix $u \in V$ and let $C_u^{(k)}$ be the expected time for $\mathcal{W}_u$ to have been within distance k of every vertex. In analogy to (33) we have

$$C_u^{(k)} \leq t + \sum_{v \in V} \sum_{s \geq t} \mathbf{Pr}(\boldsymbol{A}_s^{(k)}(v)). \tag{41}$$

where $\boldsymbol{A}_s^{(k)}(v)$ is the event that $\mathcal{W}_u$ has not been within distance k by time s.

Now fix v with $dist(u,v) > k$. Assume that v is tree-like. Define Γ_0 by contracting $M_k(v)$ to a single vertex Z and deleting any loops created (M_k is defined in Section 4). There is a natural measure preserving mapping from the set of walks in G which start at u and do not get within distance k of v to the corresponding set of walks in Γ_0 which do not visit Z. Thus the probability that $\mathcal{W}_u$ does not get within distance k in the first t steps is equal to the probability that a random walk $\widehat{\mathcal{W}}_u$ in Γ_0 which also starts at u does not visit Z in the first t steps i.e. $\mathbf{Pr}(\boldsymbol{A}_t(Z)) = \mathbf{Pr}(\boldsymbol{A}_s^{(k)}(v))$.

We apply Lemma 4 to Γ. $\pi_Z = \frac{|N_k(v)|}{rn - O(1)} = \frac{(r-1)^k}{n-O(1)}$, $R_Z \sim \frac{r-1}{r-2}$ and $H_Z/R_Z \leq 9/10$. So if now $t_0 = \lceil \frac{1+\sigma^{-1}}{(r-2)(r-1)^{k-1}} n \log n \rceil$ then $\sum_{t \geq t_0} \mathbf{Pr}(\boldsymbol{A}_t(Z)) = O(1)$. Thus

$$\sum_{v \in V_1} \sum_{t \geq t_0} \mathbf{Pr}(\boldsymbol{A}_t^{(k)}(v)) = O(n). \tag{42}$$

Now $\boldsymbol{A}_t^{(k)}(v) \subseteq \boldsymbol{A}_t(v)$ and (36) holds, even with the smaller value of t_0. Thus

$$\sum_{v \in V_2} \sum_{t \geq t_0} \mathbf{Pr}(\boldsymbol{A}_t^{(k)}(v)) = o(n) \tag{43}$$

and an upper bound of $t_0 + o(t_0)$ for $C_u^{(k)}$ follows from (41), (42) and (43).

The lower bound is obtained by taking a set S of $n^{1-o(1)}$ tree-like vertices at distance at least 3σ apart and using the Chebychev inequality as we did in Section 5.2. Choose $u \in S$ and then for each pair of vertices $v_1, v_2 \in S \setminus \{u\}$ we form Γ_1 by contracting $M_k(v_1) \cup M_k(v_2)$ into a single vertex, removing loops and then arguing as we did before. □

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