The cover time of random regular graphs

Colin Cooper*  Alan Frieze†

August 13, 2004

Abstract

Let \( r \geq 3 \) be constant, and let \( \mathcal{G}_r \) denote the set of \( r \)-regular graphs with vertex set \( V = \{1, 2, \ldots, n\} \). Let \( G \) be chosen randomly from \( \mathcal{G}_r \). We prove that \textbf{wph} the cover time of a random walk on \( G \) is asymptotic to \( \frac{r-1}{r^2} n \log n \).

1 Introduction

Let \( G = (V, E) \) be a connected graph, let \( |V| = n \), and \( |E| = m \). A \textit{random walk} \( \mathcal{W}_u \), \( u \in V \) on the undirected graph \( G = (V, E) \) is a Markov chain \( X_0 = u, X_1, \ldots, X_t, \ldots \in V \) associated to a particle that moves from vertex to vertex according to the following rule: the probability of a transition from vertex \( i \), of degree \( d_i \), to vertex \( j \) is \( 1/d_i \) if \( \{i, j\} \in E \), and 0 otherwise. For \( u \in V \) let \( C_u \) be the expected time taken for \( \mathcal{W}_u \) to visit every vertex of \( G \). The \textit{cover time} \( C_G \) of \( G \) is defined as \( C_G = \max_{u \in V} C_u \). The cover time of connected graphs has been extensively studied. It is a classic result of Aleliunas, Karp, Lipton, Lovász and Rackoff [2] that \( C_G \leq 2m(n - 1) \). It was shown by Feige [8], [9], that for any connected graph \( G \)

\[
(1 - o(1))n \log n \leq C_G \leq (1 + o(1)) \frac{4}{27} n^3.
\]

The lower bound is achieved by (for example) the complete graph \( K_n \), whose cover time is determined by the Coupon Collector problem.

In a previous paper [7] we studied the cover time of random graphs \( G_{n,p} \) when \( np = c \log n \) where \( c = O(1) \) and \( (c - 1) \log n \to \infty \). This extended a result of Jonasson, who proved in [12] that when the expected average degree \( (n - 1)p \) grows faster than \( \log n \), \textbf{wph} a random

*Department of Computer Science, King’s College, University of London, London WC2R 2LS, UK
†Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh PA15213. Supported in part by NSF grant CCR-0200945.
graph has the same cover time (asymptotically) as the complete graph $K_n$, whereas, when $np = \Omega(\log n)$ this is not the case.

(A sequence of events $\mathcal{E}_n, n \geq 0$, is said to occur with high probability (whp), if $\lim_{n \to \infty} \Pr(\mathcal{E}_n) = 1$.)

**Theorem 1.** [7] Suppose that $np = c \log n = \log n + \omega$ where $\omega = (c - 1) \log n \to \infty$ and $c \geq 1$. If $G \in G_{n,p}$, then whp

$$C_G \sim c \log \left( \frac{c}{c - 1} \right) n \log n.$$ 

The notation $A_n \sim B_n$ means that $\lim_{n \to \infty} A_n / B_n = 1$.

The main new result of the paper concerns the cover time of random regular graphs.

**Theorem 2.** Let $r \geq 3$ be constant. Let $\mathcal{G}_r$ denote the set of $r$-regular graphs with vertex set $V = \{1, 2, \ldots, n\}$. If $G$ is chosen randomly from $\mathcal{G}_r$, then whp

$$C_G \sim \frac{r - 1}{r - 2} n \log n.$$ 

Using a similar argument we can consider how many steps are needed for the walk to get within distance $k$ of every vertex. Let us call this $C_G^{(k)}$. Cover time corresponds to $k = 0$. We prove

**Theorem 3.** Let $r \geq 3$, $k \geq 0$ be constants. Let $\mathcal{G}_r$ denote the set of $r$-regular graphs with vertex set $V = \{1, 2, \ldots, n\}$. If $G$ is chosen randomly from $\mathcal{G}_r$, then whp

$$C_G^{(k)} \sim \frac{1}{(r - 2)(r - 1)^{k-1}} n \log n.$$ 

The next section contains the heart of the proof of our Theorems. In it we establish a good estimate of the probability that the first visit of $\mathcal{W}$ to a vertex $v$ takes place at a time $t$. Once this is done, we can proceed to the proof of Theorem 2 in Section 3 and the proof of Theorem 3 in Section 6.

## 2 The first visit time lemma.

### 2.1 Convergence of the random walk

In this section $G$ denotes a fixed connected graph, and $u$ is some arbitrary vertex from which a walk $\mathcal{W}_u$ is started. Let $\mathcal{W}_u(t)$ be the vertex reached at step $t$, let $P$ be the matrix of transition probabilities of the walk and let $P_u^{(t)}(v) = \Pr(\mathcal{W}_u(t) = v)$. Let $\pi_v = \frac{d_v}{2m}$ for $v \in V$. 


Let $\lambda_{\text{max}} > 0$ be the second largest absolute value of an eigenvalue of $P$. Assume that $\lambda_{\text{max}} < 1$. Then,

$$|P^{(t)}_u(x) - \pi_x| \leq (\pi_x/\pi_u)^{1/2} \lambda_{\text{max}}^t \leq n^{1/2} \lambda_{\text{max}}^t.$$  \hspace{1cm} (1)

See for example [11]. (Note that connectivity and $\lambda_{\text{max}} < 1$ implies ergodicity).

### 2.2 Generating function formulation

For the results of this section, we do not require that $G$ be regular.

Fix two vertices $u,v$. Let $h_t$ be the probability $\Pr(\mathcal{W}_u(t) = v) = P^{(t)}_u(v)$, that the walk $\mathcal{W}_u$ visits $v$ at step $t$. Let $H(s)$ be the generating function for the sequence $h_t, t \geq 0$.

Similarly, considering the walk $\mathcal{W}_v$, starting at $v$, let $r_t$ be the probability that this walk returns to $v$ at step $t = 0,1,\ldots$. Let $R(s)$ be the generating function for the sequence $r_t, t \geq 0$. We note that $r_0 = 1$.

Let $f_t(u \rightarrow v)$ be the probability that the first visit of the walk $\mathcal{W}_u$ to $v$ occurs at step $t$. If $u \neq v$ then $f_0(u \rightarrow v) = 0$. Let $F(s)$ generate $f_t(u \rightarrow v)$. Thus

$$H(s) = F(s)R(s).$$  \hspace{1cm} (2)

Let

$$T = \frac{4\log n}{\log 1/\lambda_{\text{max}}}. \hspace{1cm} (3)$$

We note that (1) gives

$$\max_{x \in V} |P^{(t)}_u(x) - \pi_x| \leq n^{-3} \hspace{1cm} \text{for } t \geq T. \hspace{1cm} (4)$$

For $R(s)$ let

$$R_T(s) = \sum_{j=0}^{T-1} r_js^j. \hspace{1cm} (5)$$

Thus $R_T(s)$ generates the probability of a return to $v$ during steps $0,\ldots,T-1$ of a walk starting at $v$. Similarly for $H(s)$, let

$$H_T(s) = \sum_{j=0}^{T-1} h_js^j. \hspace{1cm} (6)$$

### 2.3 First visit time: Single vertex $v$

The following lemma should be viewed in the context that $G$ is an $n$ vertex graph which is part of a sequence of graphs with $n$ growing to infinity. We prove it in greater generality than is needed for the proof of Theorem 2.

In what follows $c_1, c_2, \ldots$ are positive constants independent of $n$. 

...
Lemma 4. Let $T$ be as defined in (3). Suppose that

(a) $H_T(1) \leq (1 - c_1)R_T(1)$.
(b) $\max_{|s| = 1} |\frac{R_T(s) - R_T(1)}{R_T(1)}| \leq 1 - c_2$.
(c) $T\pi_v = o(1)$, $T\pi_v = \Omega(n^{-2})$.
(d) $\lambda_{\max} \leq c_3 < 1$.

Let
$$\lambda = \frac{c_2}{100T}. \quad (7)$$

Let
$$p_v = \frac{\pi_v}{R_T(1)(1 + O(T\pi_v))},$$
$$c_{u,v} = 1 - \frac{H_T(1)}{R_T(1)(1 + O(T\pi_v))}, \quad (9)$$

where the values of the $1 + O(T\pi_v)$ terms are given implicitly in (16), (19) respectively. Then
$$f_t(u \rightarrow v) = c_{u,v} \frac{p_v}{(1 + p_v)^t + 1} + O(e^{-\lambda t/2}) \quad \text{for all } t \geq T. \quad (10)$$

Proof Write

$$R(s) = R_T(s) + \widehat{R}_T(s) + \frac{\pi_v s^T}{1 - s}, \quad (11)$$
$$A(s) = (1 - s)R(s) = \pi v s^T + (1 - s)(R_T(s) + \widehat{R}_T(s)), \quad (12)$$

where $R_T(s)$ is given by (5) and

$$\widehat{R}_T(s) = \sum_{t \geq T} (r_t - \pi_v) s^t$$

generates the error in using the stationary distribution $\pi_v$ for $r_t$ when $t \geq T$.

Note that while (11) is only valid for $|s| < 1$, the fact that $|r_t - \pi_v| \leq n^{1/2}c_3^t$ means that the expansion (12) is valid for $|s| < c_3^{-1}$.

Similarly, let

$$H(s) = H_T(s) + \widehat{H}_T(s) + \pi v s^T, \quad (13)$$
$$B(s) = (1 - s)H(s) = \pi v s^T + (1 - s)(H_T(s) + \widehat{H}_T(s)). \quad (14)$$
Using (11), (13) we rewrite \( F(s) = H(s)/R(s) \) from (2) as \( F(s) = B(s)/A(s) \)
If \( |s| \leq \lambda_{\text{max}}^{-1/3} \) then (1) implies that for \( Z = H, R \),
\[
|\hat{Z}(s)| \leq n^{1/2} \sum_{t \geq T} (s\lambda_{\text{max}})^t = o(n^{-2}). \tag{15}
\]
For real \( s \geq 1 \) and \( Z = H, R \), we have
\[
Z_T(1) \leq Z_T(s) \leq Z_T(1)s^T.
\]
Let \( s = 1 + \beta \pi_v \), where \( \beta > 0 \) is constant. Since \( T \pi_v = o(1) \) we have
\[
Z_T(s) = Z_T(1)(1 + O(T \pi_v)).
\]
\( T \pi_v = o(1) \) implies that \( |s| \leq \lambda_{\text{max}}^{-1/3} \) and (15) applies. As \( T \pi_v = \Omega(n^{-2}) \) and \( R_T(1) \geq 1 + r_2 > 1 + \frac{1}{n} \) this implies that
\[
A(s) = \pi_v(1 - \beta R_T(1)(1 + O(T \pi_v))).
\]
It follows that \( A(s) \) has a real zero at \( s_0 \), where
\[
s_0 = 1 + \frac{\pi_v}{R_T(1)(1 + O(T \pi_v))} = 1 + p_v, \tag{16}
\]
say. We also see that
\[
A'(s_0) = -R_T(1)(1 + O(T \pi_v)) \neq 0 \tag{17}
\]
and thus \( s_0 \) is a simple zero (see e.g. [6] p193). The value of \( B(s) \) at \( s_0 \) is
\[
B(s_0) = \pi_v \left( 1 - \frac{H_T(1)}{R_T(1)(1 + O(T \pi_v))} + O(T \pi_v) \right) \neq 0. \tag{18}
\]
Thus, from (8), (9)
\[
\frac{B(s_0)}{A'(s_0)} = -p_v c_{u,v} \tag{19}
\]
Thus the residue of \( F(s) \) at \( s_0 \) is \( B(s_0)/A'(s_0) \) (see e.g. [6] p195) and the principal part of the
Laurent expansion of \( F(s) \) at \( s_0 \) is
\[
f(s) = \frac{B(s_0)/A'(s_0)}{s - s_0}. \tag{20}
\]
To approximate the coefficients of the generating function \( F(s) \), we now use a standard technique for the asymptotic expansion of power series (see e.g.[13] Th 5.2.1).

We prove (below) that \( s_0 \) is the only zero of \( A(s) \) inside the circle \( C_\lambda = \{ s = (1 + \lambda)e^{i\theta} \} \).
Thus \( F(s) = f(s) + g(s) \), where \( g(s) \) is analytic in \( C_\lambda \). Let \( M = \max_{s \in C_\lambda} |g(s)| \). Thus \( M \leq \max |f(s)| + \max |F(s)| \).
As $F(s) = B(s)/A(s)$ on $C_\lambda$ we have that
\[ |F(s)| \leq \frac{H_T(1 + \lambda)}{|R_T(s)| - O(T^2)}. \]
Let $\tilde{s} = s/(1 + \lambda)$. We note that $|R_T(s) - R_T(\tilde{s})| \leq ((1 + \lambda)^T - 1)R_T(1)$ and also that (b) implies that $|R_T(\tilde{s})| \geq c_2 R_T(1)$ which implies that
\[ |R_T(s)| \geq R_T(1) \left( c_2 + 1 - e^{-2/100} \right), \]
and hence $M = O(1)$.

Let $a_t = [s^t]g(s)$, then (see e.g. [6] p143), $a_t = g^{(t)}(0)/t!$. By the Cauchy Inequality (see e.g. [6] p130) we have that $|g^{(t)}(0)| \leq M t!(1 + \lambda)^t$ and thus
\[ |a_t| \leq \frac{M}{(1 + \lambda)^t} = O(e^{-t\lambda/2}). \]

As $[s^t]F(s) = [s^t]f(s) + [s^t]g(s)$ and $[s^t]1/(s - s_0) = -1/(s_0)^{t+1}$ we have
\[ [s^t]F(s) = \frac{-B(s_0)/A'(s_0)}{s_0^{t+1}} + O(e^{-t\lambda/2}). \]
Thus, we obtain
\[ [s^t]F(s) = c_{u,v} \frac{p_v}{(1 + p_v)^{t+1}} + O(e^{-t\lambda/2}), \]
which completes the proof of (10).

We now prove that $s_0$ is the only zero of $A(s)$ inside the circle $C_\lambda$. We use Rouché’s Theorem (see e.g. [6]), the statement of which is as follows: Let two functions $f(z)$ and $g(z)$ be analytic inside and on a simple closed contour $C$. Suppose that $|f(z)| > |g(z)|$ at each point of $C$, then $f(z)$ and $f(z) + g(z)$ have the same number of zeros, counting multiplicities, inside $C$.

Let the functions $f(s), g(s)$ be given by $f(s) = \pi_v s^T(1 - s_0) R_T(1)$ and $g(s) = \pi_v s^T + (1 - s_0)(R_T(s) - R_T(1) + \tilde{R}_T(s))$. For $s \in C_\lambda$, let $\tilde{s} = s/(1 + \lambda),$
\[ |g(s)|/|f(s)| \leq \frac{\pi_v(1 + \lambda)^T}{\lambda R_T(1)} + \frac{|R_T(s) - R_T(\tilde{s})|}{R_T(1)} + \frac{|R_T(\tilde{s}) - R_T(1)|}{R_T(1)} + o(n^{-2}) \]
\[ \leq 100 e^{-2/100} \pi_v T + (e^{-2/100} - 1) + (1 - c_2) + o(n^{-2}) \]
As $f(s) + g(s) = A(s)$ we conclude that $A(s)$ has only one zero inside the circle $C_\lambda$. This is the simple zero at $s_0$. \qed

**Corollary 5.** Let $A_t(v)$ be the event that $W_u$ has not visited $v$ by step $t$. Then for $t \geq T$,
\[ \Pr(A_t(v)) = \frac{c_{u,v}}{(1 + p_v)^{t+1}} + O(\lambda^{-1} e^{-\lambda/2}). \]

**Proof** We use Lemma 4 and $\Pr(A_t(v)) = \sum_{r=1}^{t} f_r(u \rightarrow v)$. \qed
3 Random regular graphs are nice

Our task now is to show that a typical $r$-regular graph satisfies the conditions (a) – (d) of Lemma 4 and to compute $R_r(1)$.

We start with some typical properties of a random regular graph. Let

$$\sigma = \lceil \log \log \log n \rceil.$$  

Say a cycle $C$ is small if $|C| \leq \sigma$.

An $r$-regular graph $G$ is nice if

P1. $G$ is connected.

P2. The second eigenvalue of the adjacency matrix of $G$ is at most $2\sqrt{r-1} + \epsilon$, where $\epsilon > 0$ is arbitrarily small ($\epsilon = 1/10$ is small enough).

P3. There are at most $r^{2\sigma}$ vertices on small cycles.

P4. No pair of small cycles are within distance $3\sigma$ of each other.

**Theorem 6.** Let $r \geq 3$ be a constant and let $G$ be chosen uniformly from the set $\mathcal{G}_r$ of $r$-regular graphs with vertex set $[n]$. Then $G$ is nice whp.

**Proof**

(P1) That a random $r$-regular graph is $r$-connected whp, for $r \geq 3$, was proved in [4].

(P2) That the second eigenvalue of a random $r$-regular graph is this small whp was proved by Friedman [10].

For P3,P4 we use the configuration model as elaborated in [5]. Let $W = [n] \times [r]$ ($W_v = v \times [r]$ represents $r$ half edges incident with vertex $v \in [n]$.) A configuration $F$ is a partition of $W$ into $rn/2$ 2-element subsets and $\Omega$ denotes the set of possible configurations. We associate with $F$ a multigraph $\mu(F) = ([n], E(F))$ where, as a multi-set,

$$E(F) = \{(v, w) : \{(v, i), (w, j)\} \in F \text{ for some } 1 \leq i, j \leq r\}.$$  

(Note that $v = w$ is possible here.)

We say that $F$ is simple if the multigraph $\mu(F)$ has no loops or multiple edges. Let $\Omega_0$ denote the set of simple configurations. It is known that if $F$ is chosen uniformly from $\Omega$ then

(a) Each $G \in \mathcal{G}(n, r)$ is the image (under $\mu$) of exactly $(r!)^n$ simple configurations.

(b) $\Pr(F \in \Omega_0) \approx e^{-(r^2-1)/4}$.  

7
It follows from this that any property of almost every $\mu(F)$ is a property of almost every member of $\mathcal{G}_r$.

(P3) The expected number of small cycles in $\mu(F)$, $F$ chosen randomly from $\Omega$, is bounded by

$$\sum_{k=3}^{\sigma} \binom{n}{k} \frac{(k-1)!}{2} \frac{(r(r-1))}{2}^k \frac{M_{r-2k}}{M_r} \leq \sum_{k=3}^{\sigma} \binom{n}{k} \frac{(k-1)!}{2} \left( \frac{r}{n} \right)^k \leq r^\sigma,$$

where $M_{2m} = \frac{(2m)!}{2^{2m} m!}$ and $|\Omega| = M_r$.

The almost sure occurrence of property P3 now follows from the Markov inequality.

(P4) Similarly, the expected number of pairs of small cycles which are close to each other is bounded by

$$\sum_{a=3}^{\sigma} \sum_{b=3}^{\sigma} \binom{n}{a} \binom{n}{b-1} \frac{(a-1)!}{2} \frac{(b-1)!}{2} \left( \frac{r}{n} \right)^{a+b} + \sum_{a=3}^{\sigma} \sum_{b=3}^{\sigma} \sum_{c=1}^{\sigma} \binom{n}{a} \binom{n}{b} \binom{n}{c} \frac{(a-1)!}{2} \frac{(b-1)!}{2} \frac{c}{ab} \left( \frac{r}{n} \right)^{a+b+c+1} = o(1).$$

\[\square\]

Remark 1. Although the main subject of the paper is random regular graphs, it is worth mentioning Ramanujan graphs. An $n$-vertex $r$-regular graph is Ramanujan if $\lambda_{\text{max}} \leq \frac{2\sqrt{r-1}}{r}$. It is known that such graphs have girth $\Omega(\log n)$ and so they are nice, see Alon [3]. Consequently, their cover time $\sim \frac{r-1}{r-2} n \log n$.

Remark 2. Aldous [1] considered the cover time of Cayley graphs and obtained a similar expression for the cover time. By relaxing the assumptions in Lemma 4 it is possible to obtain some of his results e.g. the hypercube and toroidal grids in three or more dimensions.

4 Nice graphs

Assume from now on that $G$ is a nice regular graph. For $v \in V$ and $k \geq 0$, let $N_k(v) = \{w : \text{dist}(v, w) = k\}$ be the set of vertices at distance $k$ from $v$. Let $M_t(v) = \bigcup_{j=0}^{t} N_t(v)$, and let $G_t(v)$ be the subgraph of $G$ induced by $M_t(v)$. Also let us replace the notations $R_T(1), H_T(1)$ by $R_v, H_v$ reflecting their dependence on $v$.

Definition 1. We say $v$ is locally tree-like if $G_\sigma(v)$ is a tree.

Lemma 7. If $v$ is locally tree-like then

$$R_T(1) = \frac{r-1}{r-2} + o(\sigma^{-1}).$$
Proof. Let $T_r$ be the infinite $r$-regular tree, rooted at $v$. Let $\mathcal{X}$ be a random walk on $T_r$ starting at $v$. Let $\rho_i$ be the probability that $\mathcal{X}$ is at $v$ at step $i$. Now we can project the walk $\mathcal{X}$ onto a walk $\mathcal{Y}$ on $\{0, 1, 2, \ldots, \}$ where the particle moves right with probability $q = \frac{r-1}{r}$ and left with probability $p = \frac{1}{r}$, except of course at the origin, where it must move right. Let $E_i$ be the expected number of visits to 0 for $\mathcal{Y}$ starting at $i$. Then

$$E_0 = 1 + E_1 = 1 + E_0 p/q.$$  

This is because $E_1$ is $E_0$ times the expected number of visits to 0 between right moves from 1. Solving gives

$$\sum_{i=0}^{\infty} \rho_i = E_0 = \frac{r-1}{r-2}. \quad (22)$$

Note next that for $i \geq 0$ we have $\rho_{2i+1} = 0$ and we will argue that

$$\rho_{2i} \leq \binom{2i}{i} \frac{(r-1)^i}{r^{2i}}. \quad (23)$$

and then

$$\sum_{i=\sigma+1}^{\infty} \rho_i \leq \sum_{j=\sigma/2}^{\infty} \binom{2j}{j} \frac{(r-1)^j}{r^{2j}} = o(\sigma^{-1}). \quad (24)$$

We compare this with $R_T(1)$. First observe that $r_i = \rho_i$ for $i \leq \sigma$. Then from (1) we see that

$$\sum_{i=\sigma+1}^{T} r_i \leq \sum_{i=\sigma+1}^{T} (\pi_v + \lambda_{\text{max}}^i) = o(\sigma^{-1}).$$

Let us now prove (23). First observe that the RHS of (23) is the probability that a walk $\mathcal{Y}_1$ is at the origin after $2i$ steps. Here $\mathcal{Y}_1$ is the walk on $\{0, \pm1, \pm2, \ldots, \}$ where the particle moves right with probability $q = \frac{r-1}{r}$ and left with probability $p = \frac{1}{r}$ i.e. there is no barrier at the origin. We can couple $\mathcal{Y}, \mathcal{Y}_1$ so that $\mathcal{Y}(t) \geq |\mathcal{Y}_1(t)|$. When $\mathcal{Y}_1(t) > 0$ we can move them in the same direction and when $\mathcal{Y}_1 < 0$ then we can move $\mathcal{Y}$ further from the origin whenever $\mathcal{Y}_1$ moves further from the origin.

The lemma now follows from (22) and (24).

Remark 1. Because there are very few non-tree-like vertices and because they are far apart, we will find that we do not need to estimate $R_T(1)$ for such vertices. It is relatively easy to show that for non-tree-like vertices $R_T(1) = 1 + O(r^{-1})$ as $r \to \infty$, thus the only difficulty is with small $r$.

Lemma 8. If $v$ is locally tree-like then for $|s| = 1$, $|R_T(s) - R_T(1)| \leq \frac{5}{6}$.

Proof. For any $s$,

$$|R_T(s) - R_T(1)| \leq \sum_{j=1}^{T} r_j |s^j - 1|.$$
As \( |s| = 1 \) we have that
\[
\sum_{j=1}^{T} r_j |s^j - 1| \leq 2 \sum_{j=1}^{T} r_j. \tag{25}
\]

We prove the lemma for \( r \geq 4 \) by observing that Lemma 7 implies
\[
2 \sum_{j=1}^{T} r_j = 2(R_T(1) - 1) = (1 + o(1)) \frac{2}{r - 2} \leq (1 + o(1)) \frac{2}{3} \frac{r - 1}{r - 2} = (1 + o(1)) \frac{2}{3} R_T(1). \tag{26}
\]

When \( r = 3 \) we improve on (25) using ad-hoc arguments. First observe that \( \pi_v = 1/n \) for \( v \in V \) and that (1) implies that
\[
S_0 = \sum_{i=\sigma}^{T} r_j |s^j - 1| \leq 2 \sum_{i=\sigma}^{T} r_i \leq 2 \sum_{i=\sigma}^{T} (\lambda^i_{\max} + \pi_v) = o(1). \tag{27}
\]

Now consider \( j < \sigma \). For a locally tree-like vertex, \( r_j = 0 \) if \( j \) is odd, and \( r_j > 0 \) if \( j \) is even. Fix \( 0 \leq \theta < 2\pi \) and let \( s = e^{i\theta} \), then for \( j = 2k \)
\[
|s^j - 1| = (2(1 - \cos j\theta))^{1/2} = 2|\sin k\theta|. 
\]

Thus
\[
S_1 = \sum_{j=1}^{\sigma - 1} r_j |s^j - 1| = 2 \sum_{k=1}^{[(\sigma - 1)/2]} r_{2k} |\sin k\theta|. 
\]

Note now that \( r_2 = \frac{1}{3} \) and \( r_4 = \frac{5}{27} \). Suppose first that \( \theta \notin I = [\frac{3\pi}{16}, \frac{5\pi}{16}] \cup [\frac{11\pi}{16}, \frac{13\pi}{16}] \). Then
\[
|\sin 2\theta| \leq \sin \frac{3\pi}{8} \quad \text{and so}
\]
\[
S_1 \leq 2 \sum_{j=1}^{\sigma - 1} r_j - \frac{2}{3} \left(1 - \sin \frac{3\pi}{8}\right). \tag{28}
\]

On the other hand, if \( \theta \in I \) then \( |\sin 4\theta| \leq \sin \frac{\pi}{4} \) and then
\[
S_1 \leq 2 \sum_{j=1}^{\sigma - 1} r_j - \frac{10}{27} \left(1 - \sin \frac{\pi}{4}\right). \tag{29}
\]

(27), (28), (29) imply that \( S_0 + S_1 \leq 2(R_T(1) - 1) - 1/3 \). The lemma follows, since \( R_T(1) \sim 2 \) for \( r = 3 \).

Finally we note:

**Lemma 9.** For nice graphs, \( \frac{H_T(1)}{R_T(1)} \leq \frac{9}{16} \).
Proof Let $f'_t$ be the probability that $W_u$ has a first visit to $v$ at time $t$. As $H(s) = F(s)R(s)$ we have

$$H_T(1) \leq \Pr(W_u \text{ visits } v \text{ by time } T - 1)R_T(1)$$

$$= R_T(1) \sum_{t = 1}^{T-1} f'_t.$$ 

Now (1) implies that if $\tau_0 = [2 \log \lambda_{\max}^{-1} \log \log n]$ then

$$\sum_{t = \tau_0}^{T-1} f'_t \leq \sum_{t = \tau_0}^{T-1} (\pi_v + \lambda_{\max}^t) = o(1).$$

We now estimate $\sum_{t = 0}^{\tau_0} f'_t$, the probability that $W_u$ visits $v$ by time $\tau_0$. Let $v_1, v_2, \ldots, v_r$ be the neighbours of $v$ and let $w$ be the first neighbour of $v$ visited by $W_u$. Then

$$\Pr(W_u \text{ visits } v \text{ by time } \tau_0) = \sum_{i = 1}^{r} \Pr(W_u \text{ visits } v \text{ by time } \tau_0 | w = v_i)\Pr(w = v_i)$$

$$\leq \sum_{i = 1}^{r} \Pr(W_{v_i} \text{ visits } v \text{ by the time } \tau_0)\Pr(w = v_i).$$

So it suffices to prove the lemma when $u$ is a neighbour of $v$. If $G_t(u)$ is a tree then we can argue as in Lemma 7. Let $\psi$ be the probability that a particle at the root of $T_r$ ever returns to the root. The expected number of visits is

$$\frac{r - 1}{r - 2} = \sum_{k = 1}^{\infty} k\psi^{k-1}(1 - \psi) = \frac{1}{1 - \psi}.$$ 

So $\psi = \frac{1}{r-1}$ and

$$\Pr(W_u \text{ does not visit } v \text{ by time } \tau_0) \geq \frac{r - 1}{r}(1 - \psi - o(1)) = \frac{r - 2}{r} - o(1).$$

If $G_t(u)$ contains a cycle $C$ then let $e = (\xi, \eta)$ be an edge of $C$ not incident with $u$ and let $T_u$ be the tree $G_t(u) - e$. Let $N'(u) = \{u_1, u_2, \ldots, u_s\}$, $s \in \{r - 2, r - 1\}$ be the neighbours of $u$ which are not on a shortest path from $\xi$ or $\eta$ to $u$ in $T_u$. $|N'(u) \setminus \{v\}| \geq r - 3$ and so

$$\Pr(W_u \text{ does not visit } v \text{ by time } \tau_0) \geq \frac{r - 3}{r}(1 - \psi - o(1)) = \frac{(r - 2)(r - 3)}{r(r - 1)} - o(1).$$

This leaves the case $r = 3$ and $N'(u) = \{v\}$. With probability $\frac{2}{3}$ we have $W_u(1) \neq v$. If $\xi$ or $\eta$ is reached (possibly $N(u) = \{v, \xi, \eta\}$), then with probability $\frac{1}{3}$ the next move is away from $u$ and $1 - \psi - o(1)$ bounds the probability that there is no return to $\xi$ or $\eta$. Hence

$$\Pr(W_u \text{ does not visit } v \text{ by time } \tau_0) \geq \frac{2}{9}(1 - \psi - o(1))$$

completing the proof of the lemma.  \hfill \Box
5 Cover time of nice graphs

We now prove that

$$C_G \sim \frac{r - 1}{r - 2} n \log n.$$  

Assume that $u, v \in V$ and that $v$ is tree-like. Section 3 establishes that the conditions of Lemma 4 hold, and gives values for the parameters $c_{uv}, p_v$ given by (8), (9). To summarize we have

$$R_T(1) = \frac{r - 1}{r - 2} + o(1), \quad H_T(1) \leq \frac{9}{10}, \quad \lambda_{\max} \leq \frac{2\sqrt{r - 1} + 1}{r},$$  

$$\pi_v = \frac{1}{n}, \quad T = O(\log n) \quad \lambda = \Omega(1/\log n).$$

Hence, the probability that $\mathcal{W}_u$ has not visited $v$ by some step $t \geq T$ (see Corollary 5) is given by

$$\Pr(A_t(v)) = (1 + o(1)) c_{uv} e^{-t p_v} + O(\lambda^{-1} e^{-\lambda/2}).$$

Here $c_{uv} < 1$ and

$$p_v = \frac{r - 2}{(r - 1)n} (1 + o(\sigma^{-1})).$$

5.1 Upper bound on cover time

Let $t_0 = [(1 + \sigma^{-1}) \frac{r - 1}{r - 2} n \log n]$. We prove that for nice graphs, for any vertex $u \in V$,

$$C_u \leq t_0 + o(t_0).$$  

(30)

Let $T_G(u)$ be the time taken to visit every vertex of $G$ by the random walk $\mathcal{W}_u$. Let $U_t$ be the number of vertices of $G$ which have not been visited by $\mathcal{W}_u$ at step $t$. We note the following:

$$C_u = \mathbb{E} T_G(u) = \sum_{t > 0} \Pr(T_G(u) \geq t),$$  

(31)

$$\Pr(T_G(u) > t) = \Pr(U_t > 0) \leq \min\{1, \mathbb{E} U_t\}. $$  

(32)

It follows from (31), (32) that for all $t$

$$C_u \leq t + \sum_{s \geq t} \mathbb{E} U_s = t + \sum_{v \in V} \sum_{s \geq t} \Pr(A_s(v)).$$  

(33)

Let $V_1$ be the set of locally tree-like vertices and let $V_2 = V - V_1$. If $G$ is nice then $|V_2| \leq r^{3\sigma}$ for there are at most $r^{3\sigma}$ vertices within distance $\sigma$ of a particular vertex in a small cycle, and at most $r^{2\sigma}$ vertices on small cycles.
For \( v \in V_1 \) we have

\[
\sum_{s \geq t_0} \Pr(A_s(v)) \leq (1 + o(1))e^{-t_0 p_v} \sum_{s \geq t_0} e^{-(s-t_0)p_v} + O(\lambda^{-2}e^{-\lambda t_0/2})
\]

\[
\leq 2p_v^{-1}e^{-t_0 p_v}
\]

\[
\leq 3 \frac{r - 1}{r - 2}.
\]

Furthermore, we see that in particular,

\[
\Pr(A_{5n}(v)) \leq 2e^{-1}. \tag{34}
\]

Suppose next that \( v \in V_2 \). We can find \( w \in V_1 \) such that \( \text{dist}(v, w) \leq \sigma \). So from (34), with \( \nu = 5n + \sigma \), we have

\[
\Pr(A_\nu(v)) \leq 1 - (1 - 2e^{-1})r^{-\sigma}
\]

since if our walk visits \( w \), it will with probability at least \( r^{-\sigma} \) visit \( v \) within the next \( \sigma \) steps. Thus if \( \gamma = (1 - 2e^{-1})r^{-\sigma} \),

\[
\sum_{s \geq t_0} \Pr(A_s(v)) \leq \sum_{s \geq t_0} (1 - \gamma)^{s/\nu} \tag{35}
\]

\[
\leq \sum_{s \geq t_0} (1 - \gamma)^{s/(2\nu)}
\]

\[
= \frac{(1 - \gamma)^{t_0/(2\nu)}}{1 - (1 - \gamma)^{1/(2\nu)}}
\]

\[
\leq 3\nu \gamma^{-1}. \tag{36}
\]

Thus, for all \( u \in V \),

\[
C_u \leq t_0 + 3 \frac{r - 1}{r - 2} |V_1| + 3|V_2| \nu \gamma^{-1}
\]

\[
= t_0 + O(r^{4\sigma}n)
\]

\[
= t_0 + o(t_0),
\]

as \( \sigma = \lfloor \log \log \log n \rfloor \).

### 5.2 Lower bound on cover time

For any vertex \( u \), we can find a set of vertices \( S \) such that at time \( t_1 = t_0(1 - \epsilon) \), \( \epsilon \to 0 \), the probability the set \( S \) is covered by the walk \( \mathcal{W}_u \) tends to zero. Hence \( T_G(u) > t_1 \) \textbf{whp} which implies that \( C_G \geq t_0 - o(t_0) \).

We construct \( S \) as follows. Let \( S \subseteq V_1 \) be some maximal set of locally tree-like vertices all of which are at least distance \( 2\sigma + 1 \) apart. Thus \( |S| \geq (n - r^{3\sigma})r^{-(2\sigma+1)} \).
Let $S(t)$ denote the subset of $S$ which has not been visited by $\mathcal{W}_u$ after step $t$. Now, provided $t \geq T$

$$
E |S(t)| \geq (1 - o(1)) \sum_{v \in S} \left( \frac{c_{u,v}}{(1 + p_v)^t} + o(n^{-2}) \right).
$$

Let $u$ be a fixed vertex of $S$. Let $v \in S$ and let $H_T(1)$ be given by (6), then (1) implies that

$$
H_T(1) \leq \sum_{t=0}^{T-1} (\pi_v + \lambda^t) = o(1).
$$

Thus $c_{uv} = 1 - o(1)$. Setting $t = t_1 = (1 - \epsilon) t_0$ where $\epsilon = 2 \sigma^{-1}$, we have

$$
E |S(t)| = (1 + o(1)) |S| e^{-(1-o(1)) p_v} \geq n^{1/\sigma}.
$$

Let $Y_{v,t}$ be the indicator for the event that $\mathcal{W}_u$ has not visited vertex $v$ at time $t$. Let $Z = \{v, w\} \subset S$. We will show (below) that that for $v, w \in S$

$$
E (Y_{v,t_1} Y_{w,t_1}) = \frac{c_{u,z}}{(1 + p_z)^{t_1+2}} + o(n^{-2}),
$$

where $c_{u,z} \sim 1$ and $p_z \sim 2(r-2)/(n(r-1))$. Thus

$$
E (Y_{v,t_1} Y_{w,t_1}) = (1 + o(1)) E (Y_{v,t_1}) E (Y_{w,t_1}).
$$

It follows from (38) and (40), that

$$
Pr(S(t_1) \neq 0) \geq \frac{(E |S(t_1)|)^2}{E |S(t_1)|^2} = \frac{1}{E[S_{t_1}]/(E |S(t_1)|)^2} = 1 - o(1).
$$

**Proof of (39).** Let $\Gamma$ be obtained from $G$ by merging $v, w$ into a single node $Z$. This node has degree $2r$ and every other node has degree $r$.

There is a natural measure preserving mapping from the set of walks in $G$ which start at $u$ and do not visit $v$ or $w$, to the corresponding set of walks in $\Gamma$ which do not visit $Z$. Thus the probability that $\mathcal{W}_u$ does not visit $v$ or $w$ in the first $t$ steps is equal to the probability that a random walk $\mathcal{W}_u$ in $\Gamma$ which also starts at $u$ does not visit $Z$ in the first $t$ steps.

We apply Lemma 4 to $\Gamma$. That $\pi_Z = \frac{2}{n}$ is clear, and $c_{u,z} = 1 - o(1)$ is argued as in (37). The derivation of $R_T(1)$ in Lemma 7 is also valid. The vertex $Z$ is tree-like up to distance $\sigma$ in $\Gamma$. The fact that the root vertex of the corresponding infinite tree has degree $2r$ does not affect the calculation of $R_T(1)$.

\[\square\]
6 Looking ahead

We now consider Theorem 3. Fix \( u \in V \) and let \( C^{(k)}_u \) be the expected time for \( \mathcal{W}_u \) to have been within distance \( k \) of every vertex. In analogy to (33) we have

\[
C^{(k)}_u \leq t + \sum_{v \in V} \sum_{s \geq t} \Pr(A^{(k)}_s(v)).
\]  

(41)

where \( A^{(k)}_s(v) \) is the event that \( \mathcal{W}_u \) has not been within distance \( k \) by time \( s \).

Now fix \( v \) with \( \text{dist}(u, v) > k \). Assume that \( v \) is tree-like. Define \( \Gamma_0 \) by contracting \( M_k(v) \) to a single vertex \( Z \) and deleting any loops created (\( M_k \) is defined in Section 4). There is a natural measure preserving mapping from the set of walks in \( G \) which start at \( u \) and do not get within distance \( k \) of \( v \) to the corresponding set of walks in \( \Gamma_0 \) which do not visit \( Z \). Thus the probability that \( \mathcal{W}_u \) does not get within distance \( k \) in the first \( t \) steps is equal to the probability that a random walk \( \mathcal{W}_u \) in \( \Gamma_0 \) which also starts at \( u \) does not visit \( Z \) in the first \( t \) steps i.e. \( \Pr(A_t(Z)) = \Pr(A^{(k)}_s(v)). \)

We apply Lemma 4 to \( \Gamma \). \( \pi_Z = \frac{|N_k(v)|}{|M_n(v)|} = \frac{(r-1)^k}{n^{O(1)}}, R_Z \sim \frac{r-1}{r^2} \) and \( H_Z/R_Z \leq 9/10 \). So if now \( t_0 = \lceil \frac{1+\sigma^{-1}}{(r-2)(r-1)^{k-1}} n \log n \rceil \) then \( \sum_{t \geq t_0} \Pr(A_t(Z)) = O(1). \) Thus

\[
\sum_{v \in V_1} \sum_{t \geq t_0} \Pr(A^{(k)}_t(v)) = O(n).
\]  

(42)

Now \( A^{(k)}_t(v) \subseteq A_t(v) \) and (36) holds, even with the smaller value of \( t_0 \). Thus

\[
\sum_{v \in V_2} \sum_{t \geq t_0} \Pr(A^{(k)}_t(v)) = o(n)
\]  

(43)

and an upper bound of \( t_0 + o(t_0) \) for \( C^{(k)}_u \) follows from (41), (42) and (43).

The lower bound is obtained by taking a set \( S \) of \( n^{1-o(1)} \) tree-like vertices at distance at least \( 3\sigma \) apart and using the Chebychev inequality as we did in Section 5.2. Choose \( u \in S \) and then for each pair of vertices \( v_1, v_2 \in S \setminus \{u\} \) we form \( \Gamma_1 \) by contracting \( M_k(v_1) \cup M_k(v_2) \) into a single vertex, removing loops and then arguing as we did before. \( \Box \)

References


