On the connectivity threshold for colorings of random graphs and hypergraphs

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Abstract

Let \( \Omega_q = \Omega_q(H) \) denote the set of proper \([q]\)-colorings of the hypergraph \( H \). Let \( \Gamma_q \) be the graph with vertex set \( \Omega_q \) and an edge \( \{\sigma, \tau\} \) where \( \sigma, \tau \) are colorings iff \( h(\sigma, \tau) = 1 \). Here \( h(\sigma, \tau) \) is the Hamming distance \( |\{v \in V(H) : \sigma(v) \neq \tau(v)\}| \). We show that if \( H = H_{n,m;k}, k \geq 2 \), the random \( k \)-uniform hypergraph with \( V = [n] \) and \( m = dn/k \) then w.h.p. \( \Gamma_q \) is connected if \( d \) is sufficiently large and \( q \gtrapprox (d/\log d)^{1/(k-1)} \).

1 Introduction

In this paper, we will discuss a structural property of the set \( \Omega_q \) of proper \([q]\)-colorings of the random hypergraph \( H = H_{n,m;k} \), where \( m = dn/k \) for some large constant \( d \). Here \( H \) has vertex set \( V = V(H) = [n] \) and an edge set \( E = E(H) \) consisting of \( m \) randomly chosen \( k \)-sets from \( \binom{[n]}{k} \). Note that in the graph case where \( k = 2 \) we have \( H_{n,m;2} = G_{n,m} \). A proper \([q]\)-coloring is a map \( \sigma : [n] \to [q] \) such that \( |\sigma^{-1}(e)| \geq 2 \) for all \( e \in E \) i.e. no edge is monochromatic. Then let us define \( \Gamma_q = \Gamma_q(H) \) to be the graph with vertex set \( \Omega_q \) and an edge \( \{\sigma, \tau\} \) iff \( h(\sigma, \tau) = 1 \) where \( h(\sigma, \tau) \) is the Hamming distance \( |\{v \in [n] : \sigma(v) \neq \tau(v)\}| \). In the Statistical Physics literature the definition of \( \Gamma_q \) may be that colorings \( \sigma, \tau \) are connected by an edge in \( \Gamma_q \) whenever \( h(\sigma, \tau) = o(n) \). Our theorem holds a fortiori if this is the case.

Notation: \( f(d) \gtrsim g(d) \) if \( f(d) \geq (1+\varepsilon_d)g(d) \) for \( d \) large and where \( \varepsilon_d > 0 \) and \( \lim_{d \to \infty} \varepsilon_d = 0 \). We prove the following.

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Theorem 1.1. If \( k \geq 2 \) and \( p = \frac{d}{\binom{n}{k+1}} \) and \( m = \binom{n}{k}p \) and \( q \gtrsim \left( \frac{(k-1)d}{\log d} \right)^{1/(k-1)} \) and \( d = O(1) \) is sufficiently large, then w.h.p. \( \Gamma_q \) is connected.

A more precise expression for the lower bound on \( q \) will be given in Lemma 4.1.

Note that \( \Gamma_q \) connected implies that “The Glauber Dynamics on \( \Omega_q \) is ergodic”. At the moment we only know that Glauber Dynamics is rapidly mixing for \( q \geq (1.76 \ldots)d \), see Efthymiou, Hayes, Štefankovič and Vigoda [12]. So, it would seem that the connectivity of \( \Gamma_q \) is not likely to be a barrier to randomly sampling colorings of sparse random graphs.

We note that the lower bound for \( q \) is close to where the greedy coloring algorithm succeeds w.h.p.

We should note that in the case \( k = 2 \) that Molloy [18] has shown that w.h.p. there is no giant component in \( \Gamma_q \) if \( q \lesssim \frac{d}{\log d} \). It is somewhat surprising therefore that w.h.p. \( \Gamma_q \) jumps very quickly from having no giant to being connected. One might have expected that \( q \gtrsim \frac{d}{\log d} \) would simply imply the existence of a giant component.

Prior to this paper, it was shown in [11] that w.h.p. \( \Gamma_q, q \geq d + 2 \) is connected. The diameter of the reconfiguration graph \( \Gamma_q(G) \) for graphs \( G \) has been studied in the graph theory literature, see Bousquet and Perarnau [8] and Feghali [13]. They show that if the maximum sub-graph density of a graph is at most \( d - \varepsilon \) and \( q \geq d + 1 \) then \( \Gamma_q(G) \) has polynomial diameter. Our proof for random \( G \) only shows exponential diameter but for many fewer colors. It would be interesting to study the diameter in a random setting.

Theorem 1.1 falls into the area of “Structural Properties of Solutions to Random Constraint Satisfaction Problems”. This is a growing area with connections to Computer Science and Theoretical Physics. In particular, much of the research on the graph \( \Gamma_q \) has been focused on the structure near the colorability threshold, e.g. Bapst, Coja-Oghlan, Hetterich, Rassman and Vilenchik [5], or the clustering threshold, e.g. Achlioptas, Coja-Oghlan and Ricci-Tersenghi [2], Molloy [18]. Other papers heuristically identify a sequence of phase transitions in the structure of \( H_q \), e.g., Krząkala, Montanari, Ricci-Tersenghi, Semerjian and Zdeborová [17], Zdeborová and Krząkala [20]. The existence of these transitions has been shown rigorously for some other CSPs. One of the most spectacular examples is due to Ding, Sly and Sun [10] who rigorously showed the existence of a sharp satisfiability threshold for random \( k \)-SAT.

Section 3 describes a property \((\alpha, \beta)\)-colorability such that if \( H \) has this property then \( q \geq \alpha + \beta + 1 \) implies that \( \Gamma_q \) is connected. Section 4 proves that \( H_{n,m,k}, k \geq 2 \), is \((\alpha, \beta)\)-colorable for \( \alpha, \beta \) defined in (2).

The paper uses some of the ideas from [4] which showed there is a giant component in \( \Gamma_q(G_{n,m}), m = dn/2 \) w.h.p. when \( q \geq cd/\log d \) for \( c > 3/2 \).
2 Outline argument

We identify values $\alpha \approx ((k-1)d/\log d)^{1/(k-1)} \gg \beta$ such that w.h.p. $H = H_{n,m;k}$ has the property that any greedy coloring of $H$ will need at most $\alpha$ maximal independent sets before being left with a graph without a $\beta$-core. (See Lemma 4.4.) We call the colorings found in this way, good greedy colorings and we refer to this property as $(\alpha, \beta)$-colorability. It follows from this, basically using the argument from [4], that if $\sigma \in \Omega_q$ and $q \geq \alpha + \beta$ then there is a good path in $\Gamma_q$ to some good greedy coloring $\sigma_1$.

Suppose now that $\sigma_1, \tau_1$ are good greedy colorings. If $q \geq \alpha + \beta + 1$ then there is a color $c$ that is not used by $\sigma_1$. From $\sigma_1$ we move to $\sigma_2$ by re-coloring vertices colored 1 in $\sigma_1$ by $c$. Then we move from $\sigma_2$ to $\sigma_3$ by coloring with color 1, all vertices that have color 1 in $\tau_1$. At this point, $\sigma_3$ and $\tau_1$ agree on color 1. $\sigma_3$ may use $\alpha + \beta + 1$ colors and so we move by a good path from $\sigma_3$ to a coloring $\sigma_4$ that uses at most $\alpha + \beta$ colors and does not change the color of any vertex currently with color 1. Here we use the fact that $H_{n,m;k}$ is $(\alpha, \beta)$-colorable. After this, it is induction that completes the proof.

3 $(\alpha, \beta)$-colorability

The degree of a vertex $v \in V$ in a hypergraph $H = (V,E)$ is the number of edges $e \in E$ such that $v \in e$. (For completeness, we will state several things in this short paper that one might think can be taken for granted.)

Let $H = (V,E)$. A $\beta$-core of $H$ is a maximal subgraph of $H$ in which every vertex has degree at least $\beta$. For every $U \subset V$, if the subgraph of $H$ induced by $U$ does not have a $\beta$-core then there is an ordering $\{u_1, u_2, ..., u_{|U|}\}$ of the vertices in $U$ such that every vertex in $U$ has at most $\beta - 1$ neighbors that precede it in that ordering.

If a hypergraph $H$ that does not have a $\beta$-core then we can color it with at most $\beta$ colors. Let $v_1, v_2, ..., v_n$ be an ordering on $V$ where

for every $i$, there are at most $\beta - 1$ edges that contain $v_i$

and are contained in $\{v_1, v_2, ..., v_i\}$. (1)

Such an ordering must exist when there is no $\beta$-core. We color the vertices in the order $v_1, v_2, ..., v_n$ and assign to $v_i$ a color that is not blocked by the $\beta - 1$ neighbors that precede it. A color $c$ is blocked for vertex $v$ by vertices $w_1, w_2, ..., w_{k-1}$ if $e = \{v, w_1, ..., w_{k-1}\} \in E(H)$ and $w_1, w_2, ..., w_{k-1}$ have already been given color $c$.

Next let $V_0 = \emptyset$ and let $V_1, V_2, ..., V_\alpha$ be a sequence of independent sets of $H$ such that for each $j \geq 0$, $V_{j+1}$ is maximal in the sub-hypergraph $H_j$ induced by $V \setminus \bigcup_{i \leq j} V_i$. We say that such a sequence is a maximally independent sequence of length $\alpha$. Note that we allow $V_j = \emptyset$ here, in order to make our sequences of length exactly $\alpha$. 

3
Definition 3.1. We say that a hypergraph $H$ is $(\alpha, \beta)$-colorable if there does not exist a maximally independent sequence of length $\alpha$ such that $V \setminus \bigcup_{i \leq \alpha} V_i$ has a $\beta$-core.

The main result of this section is the following.

Theorem 3.2. Let $H$ be $(\alpha, \beta)$-colorable and let $q \geq \alpha + \beta + 1$. Then $\Gamma_q(H)$ is connected.

Later, in Section 4 we will show that $H_{n,m,k}$, $k \geq 2$ is $(\alpha, \beta)$-colorable, for a suitable values of $m, \alpha, \beta$.

Lemma 3.3. Let $H = (V, E)$ be an $(\alpha, \beta)$-colorable hypergraph and $V_1 \subseteq V$ be a maximal independent set of $V$. Set $V' = V \setminus V_1$ and let $H'$ be the subgraph of $H$ induced by $V'$. Then $H'$ is $(\alpha - 1, \beta)$-colorable.

Proof. Assume that $H'$ in not $(\alpha - 1, \beta)$-colorable. Then there exists a partition of $V'$ into $V'_1, \ldots, V'_{\alpha - 1}$ such that for $j \in [\alpha - 1]$, $V'_j$ is a maximal independent set of $V' \setminus \bigcup_{\ell < j} V'_\ell$ and $W' = V' \setminus \bigcup_{\ell \leq \alpha - 1} V'_\ell$ has a $\beta$-core. For $j \in [\alpha - 1]$ set $V_{j+1} = V'_j$. Furthermore set $W = V \setminus (\bigcup V_\ell) = V' \setminus (\bigcup_{\ell < \alpha - 1} V'_\ell) = W'$. Then $V_1, \ldots, V_\alpha$ is a maximal independent sequence of length $\alpha$ and $W$ has a $\beta$-core which contradicts the fact that $H$ is $(\alpha, \beta)$-colorable. \hfill \Box

Lemma 3.4. Let $H$ be a hypergraph, $\alpha, \beta \geq 0$ and $q \geq \alpha + \beta$. Let $W \subseteq V$ be such that the subgraph of $H$ induced by $W$ has no $\beta$-core. Furthermore let $\chi$ and $\tau$ be two colorings of $H$ such that

(i) They agree on $V \setminus W$.

(ii) They use only $\alpha$ colors on the vertices in $V \setminus W$.

(iii) $\tau$ uses at most $\beta$ colors on $W$ that are distinct from the ones it uses on $V \setminus W$.

Then there exists a path from $\chi$ to $\tau$ in $\Gamma_q(H)$.

Proof. Without loss of generality we may assume that $\chi$ and $\tau$ use $[\alpha]$ to color $V \setminus W$. The proof that follows is an adaptation to hypergraphs of the proof in [4] that $\Gamma_q(G)$ is connected when a graph $G$ has no $q$-core. Because $W$ has no $\beta$-core there exists an ordering of its vertices, $v_1, v_2, \ldots, v_r$, such that for $i \in [r]$, $v_i$ has at most $\beta - 1$ neighbors in $v_1, v_2, \ldots, v_{i-1}$. For $0 \leq i \leq r$ let $\tau_i$ be the coloring that agrees with $\tau$ on $\{v_1, \ldots, v_i\}$ and with $\chi$ on $W \setminus \{v_1, \ldots, v_i\}$. On $V \setminus W$ it agrees with both. Thus $\tau_0 = \chi$ and $\tau_r = \tau$.

We proceed by induction on $i$ to show that there is a sequence of colorings $\Sigma_i$ from $\chi$ to $\tau_i$ such that (i) going from one coloring to the next in $\Sigma_i$ only re-colors one vertex and (ii) all colorings in the sequence $\Sigma_i$ are proper for the hypergraph induced by $V \setminus \{v_{i+1}, \ldots, v_r\}$. We
do not claim that the colorings in \( \Sigma_i, i < r \) are proper for \( H \). On the other hand, taking \( i = r \) we get a sequence of \( H \)-proper colorings that starts with \( \chi \), ends with \( \tau \), such that the consecutive pairs of proper colorings differ on a single vertex. Clearly, such a sequence corresponds to a path from \( \chi \) to \( \tau \) in \( \Gamma_q(H) \).

The case \( i = 1 \) is trivial as we have assumed that \( \sigma, \tau \) agree on \( V \setminus W \) and so we can give \( v_1 \) the color \( \tau(v_1) \). Assume that the assertion is true for \( i = k \geq 1 \) and let \( \chi = \psi_0, \psi_1, \ldots, \psi_s = \tau_k \) be a sequence of colorings promised by the inductive assertion. Let \( (w_j, c_j) \) denote the \((\text{vertex}, \text{color})\) change defining the change from \( \psi_{j-1} \) to \( \psi_j \). We construct a sequence of colorings of length at most \( 2s \) that yields the assertion for \( i = k + 1 \). For \( j = 1, 2, \ldots, s \), we will re-color \( w_j \) to color \( c_j \), unless there exist sets \( X_1, X_2, \ldots, X_r, r \leq \beta - 1 \) such that \((X_l \cup \{w_j\}) \subset E, 1 \leq l \leq r \) and \( \psi_{j-1}(x) = c_j, x \in \bigcup_{l=1}^{r} X_l \subset \{v_1, v_2, \ldots, v_{k+1}\} \). The bound \( r \leq \beta - 1 \) comes from our ordering. The fact that \( \psi_j \) is a proper coloring of \( V \setminus \{v_{k+1}, \ldots, v_r\} \) implies that \( v_{k+1} \in X_l, 1 \leq l \leq r \). Because \( v_{k+1} \) has at most \( \beta - 1 \) neighbors in \( \{v_1, \ldots, v_k\} \) and \( \tau \) only uses colors in \([\alpha]\) to color \( V \setminus W \), there exists a color \( c' \neq c_j \) for \( v_{k+1} \in [\alpha + \beta] \setminus [\alpha] \) which is not blocked by \( \{v_1, v_2, \ldots, v_k\} \). We first re-color \( v_{k+1} \) to \( c' \) and then we re-color \( w_j \) to \( c_j \), completing the inductive step.

\[ \text{Definition 3.5.} \quad \text{A coloring with color sets} \ V_1, V_2, \ldots, V_{\alpha+\beta} \ \text{is said to be a good greedy coloring if (i) } V_1, V_2, \ldots, V_{\alpha} \ \text{is a maximally independent sequence of length } \alpha \ \text{and (ii) } V \setminus \bigcup_{\ell \leq \alpha} V_{\ell} \ \text{has no } \beta \text{-core.} \]

We prove Theorem 3.2 in two steps. In Lemma 3.6, we show that if \( q \geq \alpha + \beta \) and \( H \) is \((\alpha, \beta)\)–colorable then we can reach a good greedy coloring in \( \Gamma_q(H) \) starting from any coloring. Then in Lemma 3.7, we show that if \( q \geq \alpha + \beta + 1 \) then any good greedy coloring \( \tau \) can be reached in \( \Gamma_q(H) \) from any other good greedy coloring \( \sigma \).

\[ \text{Lemma 3.6.} \ \text{Let } H \ \text{be an } (\alpha, \beta)\text{-colorable hypergraph, } q \geq \alpha + \beta \ \text{and } \chi \text{ be a } [q]\text{-coloring of } H. \ \text{Then there exists a good greedy coloring } \tau \text{ of } H \ \text{such that there exists a path in } \Gamma_q(H) \ \text{from } \chi \text{ to } \tau. \]

\[ \text{Proof.} \ \text{We generate the coloring } \tau \text{ as follows. Let } C_1, C_2, \ldots, C_q \ \text{be the color classes of } \chi. \ \text{Then let } V_1 \supseteq C_1 \ \text{be a maximal independent set containing } C_1. \ \text{In general, having defined } \ V_1, V_2, \ldots, V_{\ell-1} \ \text{we let } V_{\ell} = \bigcup_{1 \leq i < \ell} V_i \ \text{and then we let } V_{\ell} \ \text{be a maximal independent set in } V \setminus V_{\ell} \ \text{that contains } C_\ell \setminus V_{\ell}. \ \text{Thus } V_1, V_2, \ldots, V_{\alpha} \ \text{is a maximal independent sequence of length } \alpha. \ \text{We now describe how we transform the coloring } \chi \text{ vertex by vertex into a coloring } \chi' \text{ in which vertices in } V_i \ \text{get color } i \ \text{for } 1 \leq i \leq \alpha. \ \text{We first re-color the vertices in } V_1 \setminus C_1 \ \text{by giving them color 1, one vertex at a time. The coloring stays proper, as } V_1 \ \text{is an independent set. In general, having re-colored } V_1, V_2, \ldots, V_{\ell-1} \ \text{we re-color the vertices in } V_\ell \setminus C_\ell \ \text{with color } \ell. \ \text{Again, the coloring stays proper, as } V_\ell \ \text{is an independent set, containing all vertices in } C_\ell \ \text{that have not been re-colored. We observe that each re-coloring of a vertex } v \ \text{done while turning } \chi \text{ into } \chi' \text{ can be interpreted as moving from a coloring in } \Gamma_q(H) \ \text{to a neighboring coloring.} \]
Let \( W = V \setminus \bigcup_{1 \leq i < \ell} V_i \). Because \( H \) is \((\alpha, \beta)\)-colorable, we find that \( W \) has no \( \beta \)-core. Because \( W \) has no \( \beta \)-core there exists a proper coloring \( \tau' \) of the subgraph of \( H \) induced by \( W \) that uses only colors in \([\alpha + \beta] \setminus [\alpha] \). Set \( \tau \) to be the coloring that agrees with \( \chi' \) on \( W \setminus W \) and with \( \tau' \) on \( W \).

Lemma 3.4 implies that there is a path from \( \chi' \) to \( \tau \). Hence there is a path from \( \chi \) to \( \tau \). \( \Box \)

**Lemma 3.7.** Let \( H \) be an \((\alpha, \beta)\)-colorable hypergraph, \( q \geq \alpha + \beta + 1 \) and let \( \chi, \tau \) be two good greedy colorings. Then there exists a path from \( \chi \) to \( \tau \) in \( \Gamma^q_H \).

**Proof.** We proceed by induction on \( \alpha \). For \( \alpha = 0 \), \( H \) is \((0, \beta)\) colorable and so it does not have a \( \beta \)-core. Thus the base case follows directly from Lemma 3.4 by taking \( W = V \).

Assume that the statement of the Lemma is true for \( \alpha = k - 1 \) and let \( \alpha = k \). There exists a maximal independent sequence \( V_1, V_2, \ldots, V_k \) of length \( k \) such that if \( V' = V \setminus \bigcup_{i=1}^{k} V_i \) then (i) for \( i \in [k] \), \( \tau \) assigns the color \( i \) to \( v \in V_i \) and (ii) \( \tau \) assigns only colors in \([k + \beta] \setminus [k] \) to vertices in \( V' \).

Let \( c \) be a color not assigned by \( \chi \). There is one as \( q \geq k + \beta + 1 \). Starting from \( \chi \) we recolor all vertices that are colored 1 by color \( c \) to create a coloring \( \bar{\chi} \). Then we continue from \( \bar{\chi} \) by recoloring all the vertices in \( V_i \) by color 1 and we let \( \chi' \) be the resulting coloring. Clearly there is a path \( P_1 \) from \( \chi \) to \( \chi' \) in \( \Gamma^q_H \).

We now set \( H_1 = H \setminus V_1 \), and set \( \chi', \tau_1 \) to be the restrictions of \( \chi', \tau \) on \( H_1 \). Observe that since \( V_1 \) is a maximal independent set, Lemma 3.3 implies that \( H_1 \) is \((k - 1, \beta)\) colorable and in addition that \( \tau_1 \) is a good greedy coloring of \( H_1 \). Lemma 3.6 implies that in \( \Gamma^q_{q - 1}(H_1) \) there is a path \( P_3 \) from \( \chi_1 \) to some good greedy coloring \( \chi_1 \) that uses only \( k - 1 + \beta \) colors from \([q] \setminus \{1\} \). The induction hypothesis implies that in \( \Gamma^q_{q - 1}(H_1) \) that there is a path \( P_3 \) from \( \chi_1 \) to \( \tau_1 \).

Color 1 is not used in \( \chi_1', \tau_1 \) or in any of colorings found in the path \( P_2, P_3 \). Thus the path \( P_2, P_3 \) corresponds to a path \( P_4 \) in \( \Gamma^q_H \) from \( \chi' \) to \( \tau \). Consequently the colorings \( \chi, \tau \) are connected in \( \Gamma^q_H \) by the path \( P_1 + P_4 \). \( \Box \)

**Proof of Theorem 3.2:** Let \( H \) be \((\alpha, \beta)\) colorable, \( q \geq \alpha + \beta + 1 \), and let \( \chi_1, \chi_2 \) be two colorings of \( H \). Lemma 3.6 implies that in \( \Gamma^q_H \), there is a path \( P_i \) from \( \chi_i \) to a good greedy coloring \( \tau_i \) for \( i = 1, 2 \). Lemma 3.7 implies that there is a path in \( \Gamma^q_H \) from \( \tau_1 \) to \( \tau_2 \). \( \Box \)

## 4 Random Hypergraphs

Let

\[
\alpha = \left( \frac{(k - 1)d}{\log d - 5(k - 1) \log \log d} \right)^{\frac{1}{k - 1}}, \quad \beta = 3 \log^3 k \cdot d. \tag{2}
\]
Theorem 1.1 follows from

**Lemma 4.1.** Let $k \geq 2$ and suppose that $q \geq \alpha + \beta + 1$ and that $d$ is sufficiently large. If $p = \frac{d}{n-k-1}$ and $m = \binom{n}{k} p$ then w.h.p. $\Gamma_q(H_{n,m,k})$ is connected.

In the following we will assume for simplicity of notation that $d = O(1)$, so that $O(f(d)/n) = O(1/n)$. We do not know if there is an upper bound needed for the growth rate of $d$, but we doubt it.

To prove Lemma 4.1 we use Lemmas 4.2, 4.4, 4.5 (below) in order to deduce that w.h.p. $G_n,dn/2$ is $(\alpha, \beta)$ colorable. Then we apply Theorem 3.2. (Lemmas 4.2 and 4.5 are hardly new or best possible, but we prove them here for completeness.)

We will do our calculations on the random graph $H = H_{n,p,k} = d/\{n-k-1\}$ and use the fact for any hypergraph property $P$, we have

$$ \Pr(H_{n,m,k} \in P) \leq O(m^{1/2}) \Pr(H_{n,p,k} \in P). \quad (3) $$

**Lemma 4.2.** Let $p = \frac{d}{n-k-1}$ and $k \geq 2$ and $d$ sufficiently large. Then, w.h.p. $H = H_{n,p,k}$ does not contain an independent set of size $u = (\frac{2k \log d}{(k-1)d})^{1/(k-1)} n$.

**Proof.** Let $u = (\frac{2k \log d}{(k-1)d})^{1/(k-1)} n$. The probability that there exists an independent set of size $u$ in $H$ is bounded by

$$ \binom{n}{u} (1-p)^u \leq \left( \frac{en}{u} \right)^u \exp \left\{ -\frac{d}{n-k-1} \cdot \binom{u}{k} \right\} $$

$$ \leq \left( \frac{en}{u} \right)^u \exp \left\{ -\frac{du}{k} \cdot \frac{u}{n} \right\} \leq \frac{1}{k} \left( 1 + O \left( \frac{1}{n} \right) \right) $$

$$ = \left( e^{k-1} \frac{(k-1)d}{2k \log d} \cdot \exp \left\{ -2 \log d \left( 1 + O \left( \frac{1}{n} \right) \right) \right\} \right)^{u/(k-1)} $$

$$ = \left( e^{k-1} \frac{(k-1)}{2k \log d} \left( 1 + O \left( \frac{1}{n} \right) \right) \right)^{u/(k-1)} $$

$$ = o(1). \quad (4) $$

**Notation 4.3.** We let $m_0 = \frac{n}{\alpha}$ and $n_0 = \frac{\log^3 d}{d^{1/(k-1)} n}$. Furthermore, for $t \leq d$ we let

$$ S_t = \left\{ (s_1, s_2, ..., s_t) \in \left[ \left( \frac{2k \log d}{(k-1)d} \right)^{1/(k-1)} n \right]^t : \sum_{j=1}^t s_j \leq \min \{ tm_0, n - n_0 \} \right\}. $$
Lemma 4.4. If \( k \geq 2 \) and \( d \) is sufficiently large then, w.h.p. there does not exist \( 1 \leq t \leq d \) and disjoint sets \( V_1, \ldots, V_t \subset V \) such that:

(i) For \( i \in [t] \), \( V_i \) is a maximal independent set of the hypergraph induced by \( H = H_{n,p,k} \) on \( V \setminus (V_1 \cup \ldots \cup V_{i-1}) \),

(ii) \( (|V_1|, |V_2|, \ldots, |V_t|) \in S_t \).

Proof. Fix \( t \in [d] \), \( (s_1, \ldots, s_t) \in S_t \) and let \( \bar{s} = \frac{1}{t} \sum_{i \in [t]} s_i \). Since \( (s_1, \ldots, s_t) \in S_t \) we have that \( \bar{s} \leq \frac{1}{t} \cdot tm_0 = m_0 \). There are \( \binom{n}{s_1, s_2, \ldots, s_t, n-ts} \) ways to pick disjoint sets \( V_1, V_2, \ldots, V_t \subseteq V \) of sizes \( s_1, \ldots, s_t \) respectively. So \( V_1, \ldots, V_t \) satisfy condition (i) of Lemma 4.4 only if for every \( i \in [t] \) and every \( v \in V \setminus \bigcup_{j \in [t]} V_j \), there exist \( u_1, \ldots, u_{k-1} \in V_i \) such that \( \{u_1, \ldots, u_{k-1}, v\} \in E(H) \). So, given \( V_1, \ldots, V_t \) the probability that we have (i) is at most

\[
p_1 = \prod_{i=1}^{t} (1 - (1 - p)^{(s_i/p)\binom{n}{s_i}} \binom{n - \sum_{j=1}^{i} s_j}) \leq \exp \left\{ - \sum_{i=1}^{t} \left( (1 - p)^{(s_i/p)\binom{n}{s_i}} \left( n - \sum_{j=1}^{i} s_j \right) \right) \right\}. \tag{5}\]

Now let \( t' = \max \left\{ i : \sum_{j \leq i} s_j \leq \frac{n}{\log^2 d} \right\} \) and set \( \bar{s}' = \frac{1}{t'} \sum_{i=1}^{t'} s_i \). We consider 2 cases.

**Case 1:** \( t' \geq (1 - \frac{1}{\log d}) t \).

Now \( t\bar{s} \geq t's' \) and so \( \bar{s}' - \bar{s} \leq \frac{t't's'}{t} \leq \frac{t's'}{\log d} \), which implies that \( \bar{s}' \leq \bar{s} \left( 1 - \frac{1}{\log d} \right)^{-1} \leq m_0 \left( 1 + \frac{2}{\log d} \right) \). Then,

\[
\sum_{i=1}^{t} \left( (1 - p)^{(s_i/p)\binom{n}{s_i}} \left( n - \sum_{j=1}^{i} s_j \right) \right) \\
\geq \sum_{i=1}^{t'} \left( (1 - p)^{(s_i/p)\binom{n}{s_i}} \left( n - \sum_{j=1}^{i} s_j \right) \right) \\
\geq \frac{n}{\log^2 d} \sum_{i=1}^{t'} (1 - p)^{(s_i/p)\binom{n}{s_i}} \geq \frac{nt'}{\log^2 d} (1 - p)^{(s_i/p)\binom{n}{s_i}} \geq \frac{nt}{2 \log^2 d} (1 - p)^{(m_0(1 + \frac{2}{\log d}) \binom{n}{s_i})} \\
\geq \frac{nt}{2 \log^2 d} \exp \left\{ -(p + p^2) \left( \frac{\log d - 5(k-1) \log \log d}{(k-1)d} \right)^{1/(k-1)} \left( 1 + \frac{2}{\log d} \right)^n \right\} \\
\geq \frac{nt}{2 \log^2 d} \exp \left\{ -\log d - 5(k-1) \log \log d \cdot \left( 1 + \frac{3(k-1)}{\log d} \right) \right\} \\
\geq \frac{nt \log^2 d}{(d/k-1)}.
\]

Now

\[
\binom{n}{s_1, \ldots, s_t, n-t\bar{s}} \leq \binom{n}{\bar{s}, \ldots, \bar{s}, n-t\bar{s}} \leq \prod_{i=1}^{t} \binom{n}{\bar{s}} \leq \left( \frac{en}{\bar{s}} \right)^{t\bar{s}} \leq \left( \frac{en}{m_0} \right)^{tm_0}.
\]

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Thus the probability that for some $t \leq d$ there exist $V_1, ..., V_t$ satisfying conditions (i), (ii) of Lemma 4.4 is bounded by

$$
\sum_{t=1}^{d} \sum_{(s_1, ..., s_t) \in S_t} \left( \frac{n}{s_1, s_2, ..., s_t, n - \sum_{i \in \{t\}} s_i} \right) p_1
$$

$$
\leq \sum_{t=1}^{d} \sum_{(s_1, ..., s_t) \in S_t} \left( \frac{en}{m_0} \right)^{t m_0} \exp \left\{ -\frac{nt \log^2 d}{d^{1/(k-1)}} \right\} \leq \sum_{t=1}^{d} n^t \left( \frac{(e \alpha) (\log d)^{1/(k-1)}}{d \log d} \right)^{nt / d^{1/(k-1)}} = o(1).
$$

**Case 2:** $t' < \left( 1 - \frac{1}{\log d} \right) t$.

Thus $t - t' \geq \frac{1}{\log d}$. Observe that from Lemma 4.2 we can assume that

$$
t \geq t' \geq \left( \left( 1 - \frac{1}{\log^2 d} \right) / \left( \frac{2k \log d}{(k-1)d} \right)^{\frac{1}{k-1}} \right) - 1 \geq \frac{1}{4} \left( \frac{d}{\log d} \right)^{\frac{1}{k-1}}.
$$

For (6) we are using Lemma 4.2 to argue that we need at least this many independent sets to partition a set of size $n \left( 1 - \frac{1}{\log^2 d} \right)$.

$$
u = \frac{1}{t - t'} \sum_{i=t' + 1}^{t} s_i \leq \frac{\log d}{t} \cdot \frac{n}{\log^2 d} \leq 4 \left( \frac{\log d}{d} \right)^{\frac{1}{k-1}} \cdot \frac{n}{\log d},
$$

and now with $p_1$ as defined in (5) we have

$$
p_1 \leq \prod_{i=t' + 1}^{t} \left( 1 - (1 - p) \left( \frac{k}{k-1} \right)^{n - \sum_{j=1}^{i-1} s_j} \right) \leq \prod_{i=t' + 1}^{t} \left( 1 - (1 - p) \left( \frac{k}{k-1} \right)^{n_0} \right)
$$

$$
\leq \exp \left\{ -n_0 \sum_{i=t' + 1}^{t} (1 - p) \left( \frac{k}{k-1} \right)^{n_0} \right\}
$$

$$
\leq \exp \left\{ -n_0 (t - t') \exp \left\{ -(p + p^2) \left( \frac{u}{k - 1} \right) \right\} \right\}
$$

$$
\leq \exp \left\{ -n_0 (t - t') \exp \left\{ -d \left( \frac{u}{n} \right)^{k-1} \right\} \left( 1 + O \left( \frac{1}{n} \right) \right) \right\}
$$

$$
\leq \exp \left\{ -n_0 (t - t') \exp \left\{ -\frac{4k}{\log k-2 d} \right\} \right\}
$$

$$
\leq e^{-(t-t')n_0/2}.
$$

Thus the probability that for some $t \leq d$ there exist $V_1, ..., V_t$ satisfying conditions (i), (ii) of Lemma 4.4 is bounded by

$$
P = \sum_{t=1}^{d} \sum_{(s_1, ..., s_t) \in S_t} \prod_{i=t' + 1}^{t} \left( n - \sum_{j=1}^{i-1} s_j \right) \prod_{i=t' + 1}^{t} \left( n - \sum_{j=1}^{i-1} s_j \right) p_1
$$
Furthermore, Lemma 4.2 implies that \( \bar{\alpha} \) for sufficiently large \( n \) and we also have that \( n_0 \geq 16m_0 \log^2 d \). Therefore

\[
\left( \frac{en}{u} \right)^{(t-t')u} e^{-(t-t')n_0/4} \leq \left( \frac{en}{m_0} \right)^{(t-t')m_0} e^{-4(t-t')m_0 \log^2 d} \leq e^{-3(t-t')m_0 \log^2 d} \leq e^{-3m_0 \log d}.
\]

Furthermore, Lemma 4.2 implies that \( \bar{s}' \leq \left( \frac{2k \log d}{(k-1)d} \right)^{k-1} n \leq 3m_0 \). Thus

\[
\left( \frac{en}{s'} \right)^{s' (t-t')n_0/4} \leq \left( \frac{en}{3m_0} \right)^{3m_0} e^{-4(t-t')m_0 \log^2 d} \leq \left( \frac{en}{3m_0} \right)^{3m_0} e^{-4m_0 \log d} \leq e^{-tm_0 \log d}.
\]

So,

\[
P \leq d n^d e^{-4m_0 \log d} = o(1).
\]

\[\square\]

**Lemma 4.5.** If \( k \geq 2 \) and \( d \) is sufficiently large then w.h.p. every set \( S \subset V \) of size at most \( n_0 \) spans fewer than \( 3|S| \log^{3k} d \) edges in \( H \). Hence no subset of size at most \( n_0 \) contains a \( 3 \log^{3k} d \) core.

**Proof.** Let \( L = 3 \log^{3k} d \). The probability that there exists \( S \subset V \) of size at most \( n_0 \) that spans at least \( t = L|S| \) edges is bounded by

\[
\sum_{s=1}^{n_0} \binom{n}{s} \binom{(s)}{t} p^t \leq \sum_{s=1}^{n_0} \left( \frac{en}{s} \right)^{s} \frac{e^{s}}{t} \frac{d}{(k-1)} \left( \frac{en}{n} \right)^{k-1} \leq \sum_{s=1}^{n_0} \left( \frac{en}{s} \right)^{1/L} e^{s} \frac{d}{t} \left( \frac{s}{n} \right)^{k-1} \leq \sum_{s=1}^{n_0} \left( \frac{s}{n} \right)^{k-1-1/L} e^{1+1/L} d^t = o(1).
\]

\[\square\]

**Proof of Theorem 1.1:** Let \( \alpha, \beta \) be as in (2). We argue next that the properties given by Lemmas 4.2, 4.4 and 4.5 imply that \( H_{n,k;k} \) is \((\alpha, \beta)\)-colorable for \( d \) sufficiently large. Lemma 4.1 then follows directly from (3) and Theorem 3.2.

Consider a sequence of sets \( V_1, V_2, \ldots, V_\alpha \) such that \( V_i \) is maximally independent in \( [n] \setminus \bigcup_{j<i} V_j \) for \( j \leq \alpha \) (some of these sets can be empty). It follows from Lemma 4.4 that because \( \alpha m_0 = n \), we must have \( \sum_{i=1}^{\alpha} |V_i| \geq n - n_0 \) and then Lemma 4.5 implies that \( [n] \setminus \bigcup_{i \leq \alpha} V_i \) does not have a \( \beta \)-core. \[\square\]
References


