Square of Hamilton cycle in a random graph

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Abstract

We show that $p = \sqrt{\frac{e}{n}}$ is a sharp threshold for the random graph $G_{n,p}$ to contain the square of a Hamilton cycle. This improves the previous results of Kühn and Osthus and also Nenadov and Škorić.

1 Introduction

In this note we only consider binomial random graphs. The *binomial random graph* $G_{n,p}$ is the random graph G with vertex set $[n]$ in which every pair $\{i,j\} \in \binom{[n]}{2}$ appears independently as an edge in G with probability p. We say that a sequence of events $\mathcal{E}_n$ in a probability space holds *with high probability* (or *w.h.p.*) if the probability that $\mathcal{E}_n$ holds tends to 1 as $n \to \infty$. Throughout this note all logarithms are natural (base e) and all asymptotics are taken in n.

By the *kth power of a Hamilton cycle*, we mean a permutation $v_1, v_2, \ldots, v_n$ of $[n]$ such that $\{v_i, v_j\} \in E(G)$ whenever $i < j \le i+k$. (Here $i+k$ is to be taken as $i+k-n$ if $i+k \ge n+1$.) Hamilton cycles have long been studied in the context of random graphs (see, e.g., [1, 2, 5, 9]). Powers of Hamilton cycles are less well-studied and much less is known about them. Since the kth power of a Hamilton cycle contains kn edges, we can see that if Y_k denotes the number of copies of such, then by using Stirling's formula

$$\mathbf{E}(Y_k) = \tfrac{1}{2}(n-1)!p^{kn} \le \tfrac{1}{2}\sqrt{2\pi n}\left(\frac{n}{e}\right)^n p^{kn} = \tfrac{1}{2}\sqrt{2\pi n}\left(\frac{np^k}{e}\right)^n \tag{1}$$

and so $\mathbf{E}(Y_k) \to 0$ if $p \le \left(\frac{(1-\varepsilon)e}{n}\right)^{1/k}$ for any constant $\varepsilon > 0$. Thus, if $p \le \left(\frac{(1-\varepsilon)e}{n}\right)^{1/k}$, then w.h.p. $G_{n,p}$ contains no kth power of a Hamilton cycle.

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Kühn and Osthus [6] observed that for $k \geq 3$, $p \approx \left(\frac{e}{n}\right)^{1/k}$ is the correct threshold for the existence of the kth power of a Hamilton cycle. This comes directly from a result of Riordan [10]. For $k = 2$ they gave a bound of $p \geq n^{-1/2+\varepsilon}$ (for any $\varepsilon > 0$) being sufficient for the existence of the square of a Hamilton cycle w.h.p.. This result was improved by Nenadov and Škorić [8] to $p \geq \frac{C \log^4 n}{\sqrt{n}}$ (C is a positive constant) being sufficient for the existence of the square of a Hamilton cycle.

Here we prove a tight result.

Theorem 1 *Suppose that $np^2 = \alpha$, where $\alpha > 0$ is a constant.*

(i) If $\alpha < e$ then w.h.p. $G_{n,p}$ does not contain the square of a Hamilton cycle.

(ii) If $\alpha > e$ then w.h.p. $G_{n,p}$ contains the square of a Hamilton cycle.

The proof is based on a delicate application of the second moment method.

2 Proof

Part (i) immediately follows from (1).

The remaining part (ii) will follow from the second moment method.

We say that a permutation π of $[n]$ in $G_{n,p}$ is *square inducing* if $G_{n,p}$ contains edges $\{i, \pi(i)\}$ and $\{i, \pi^2(i)\}$ for each $i \in [n]$. For $i \in [n]$, let $d(i, \pi)$ be the number of vertices j such that either (a) $\{i, \pi(i), j\}$ or (b) $\{i, \pi^2(i), j\}$ forms a triangle.

Set

$$\gamma = 10\alpha + 100 \log n.$$

We say that a square inducing permutation π is *good* if $d(i, \pi) \leq \gamma$ for all $i \in [n]$.

Let X be the random variable that counts the number of good permutations.

Observe next that the Chernoff bounds imply that

$$\mathbf{Pr}(d(i, \pi) \geq \gamma \mid \pi) \leq 2\mathbf{Pr}(Bin(n, p^2) \geq \gamma) \leq 2\left(\frac{e\alpha}{\gamma}\right)^{\gamma} = o(n^{-1}),$$

(for the inequality see, e.g., Theorem 21.9 in [4]).

First we show that $\mathbf{E}(X) \to \infty$. The FKG inequality (see, e.g., Theorem 21.5 in [4]) implies that

$$\mathbf{E}(X) \geq n! p^{2n} (1 - o(n^{-1}))^n \approx n! p^{2n} \approx \sqrt{2\pi n}\left(\frac{np^2}{e}\right)^n = \sqrt{2\pi n}\left(\frac{\alpha}{e}\right)^n,$$

which clearly goes to infinity.

Now we have a choice. We can condition on all square inducing permutations being good or just counting good permutations. The computations are the same and we have plumped for the latter.

In the renaming part of the proof we show that $\mathbf{Pr}(X = 0) = o(1)$ by using the Chebyshev inequality.

Fix a good permutation π. Let $H(\pi) = (1, \pi(1), \pi^2(1), \ldots, \pi^{n-1}(1), 1)$ be the Hamilton cycle induced by π. (The edge set of $H(\pi)$ is $E(H) = \{\{i, \pi(i)\} : i \in [n]\}$.) Then let $N(a, b, c)$ be the number of good permutations $\widehat{\pi}$ such that:

(a) $|E(H(\pi)) \cap E(H(\widehat{\pi}))| = b$,

(b) $E(H(\pi)) \cap E(H(\widehat{\pi}))$ consists of a vertex-disjoint paths, say $P_1, P_2, \ldots, P_a$, and

(c) there are exactly c edges of the form $\{i, \pi^2(i)\}$ in the square of a Hamilton cycle induced by $\widehat{\pi}$ which are not in the square of P_j for any $j \in [a]$.

Observe that $N(a, b, c)$ does not depend on π and $0 \le c \le n - (b - a)$.

Note that

$$\frac{\mathbf{E}(X^2)}{\mathbf{E}(X)^2} \le \frac{n! N(0,0,0) p^{4n}}{\mathbf{E}(X)^2} + \sum_{b=1}^{n} \sum_{a=1}^{b} \sum_{c=0}^{n-(b-a)} \frac{n! N(a,b,c) p^{4n-(2b+c-a)}}{\mathbf{E}(X)^2}.$$

Since trivially, $N(0, 0, 0) \le n!$, we obtain,

$$\frac{\mathbf{E}(X^2)}{\mathbf{E}(X)^2} \lesssim 1 + \sum_{b=1}^{n} \sum_{a=1}^{b} \sum_{c=0}^{n-(b-a)} \frac{N(a,b,c) p^{2n-(2b+c-a)}}{\mathbf{E}(X)}. \tag{2}$$

We will show that the latter is o(1). Consequently, the Chebyshev inequality implies that

$$\mathbf{Pr}(X = 0) \le \frac{\mathbf{E}(X^2)}{\mathbf{E}(X)^2} - 1 = o(1),$$

as required.

It remains to show that the triple summation in (2) is $o(1)$. First we find an upper bound on $N(a, b, 0)$ and then we make corrections for the case $c > 0$.

Choose a vertices v_i, $1 \le i \le a$, on π. We have at most

$$n^a$$

choices. Let

$$b_1 + b_2 + \cdots + b_a = b,$$

where $b_i \ge 1$ is an integer for every $1 \le i \le a$. Note that this equation has exactly

$$\binom{b-1}{a-1}$$

solutions. For every i, we choose a path of length b_i in $H(\pi)$ which starts at v_i.

Thus, by the above consideration we can find a vertex-disjoint paths in $H(\pi)$ with the total of b edges in at most

$$n^a \binom{b-1}{a-1} \tag{3}$$

many ways.

Let $P_1, P_2, \ldots, P_a$ be any collection of the above a paths. Now we count the number of permutations $\widehat{\pi}$ containing these paths. We see each edge of P_i in at most 2 orders. Crudely, every such sequence can be chosen in at most

$$2^a$$

ways.

Now we bound the number of permutations containing these sequences. First note that

$$|V(P_i)| = b_i + 1.$$

Thus we have

$$n - \sum_{i=1}^{a} (b_i + 1) = n - b - a$$

vertices not in $V(P_1) \cup \cdots \cup V(P_a)$. We choose a permutation σ of $V \setminus (V(P_1) \cup \cdots \cup V(P_a))$. Here we have at most

$$(n-b-a)!$$

choices. Consequently, the number of permutations containing $P_1, P_2, \ldots, P_a$ is smaller than

$$(2\gamma)^a (n-b-a)!. \tag{4}$$

The factor γ^a bounds the number of choices for $\widehat{\pi}(j)$, when j is the end of a path.

Thus, by (3) and (4) and the Stirling formula we obtain

$$N(a,b,0) \le (2\gamma n)^a \binom{b-1}{a-1} (n-b-a)! \lesssim (2\gamma n)^a \binom{b-1}{a-1} \sqrt{2\pi n} \left(\frac{n}{e}\right)^{n-b-a}.$$

Since

$$\mathbf{E}(X) \gtrsim n! p^{2n} \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n p^{2n},$$

we get

$$\frac{N(a,b,0) p^{2n-(2b-a)}}{\mathbf{E}(X)} \lesssim (2\gamma n)^a \binom{b-1}{a-1} \left(\frac{e}{n}\right)^{b+a} p^{a-2b} = \left(\frac{e}{\alpha}\right)^b \binom{b-1}{a-1} (2\gamma e p)^a.$$

Hence,

$$\begin{aligned} \sum_{b=1}^{n} \sum_{a=1}^{b} \frac{N(a,b,0) p^{2n-(2b-a)}}{\mathbf{E}(X)} &\lesssim \sum_{b=1}^{n} \sum_{a=1}^{b} \left(\frac{e}{\alpha}\right)^b \binom{b-1}{a-1} (2\gamma e p)^a \\ &\lesssim 2\gamma e p \sum_{b=1}^{\infty} \left(\frac{e}{\alpha}(1 + 2\gamma e p)\right)^b = o(1), \end{aligned}$$

since $\sum_{a=1}^{b}\binom{b-1}{a-1}(2\gamma ep)^{a-1} = (1+2\gamma ep)^{b-1}$.

We now deal with the case $c > 0$.

We can account for this by replacing (4) by

$$\binom{n-b-a}{c}(2\gamma)^a\gamma^c(n-b-a-2c)!.$$

This is because each i contributing to the count c will reduce the number of choices for $\widehat{\pi}(i)$ either when (a) $\widehat{\pi}^2(i)) = \pi(i)$ or when (b) $\widehat{\pi}(i) = \pi^2(i)$ in the alternative case. We have coordinated (a), (b) here with (a), (b) in the definition of $d(i,\pi)$. In case (a) $j = \widehat{\pi}(i)$ and in case (b) $j = \pi(i)$. Note that it is not possible for i to contribute to both cases. There seems to be the possibility that there can be overlap in that we can have $\widehat{\pi}(i) = \pi^2(i)$ and $\pi(i) = \widehat{\pi}^2(i)$ but this case will correspond to a path of length one where $\pi, \widehat{\pi}$ traverse the edge $\{\pi(i), \pi^2(i)\}$ in opposite directions. Thus this will not contribute to the count c, but instead to a and b. The binomial coefficient comes from choosing the set of indices i.

It follows from the above that

$$N(a,b,c) \le N(a,b,0)\cdot\frac{(2\gamma)^a\gamma^c(n-b-a-2c)!}{c!(n-b-a-c)!}.$$

Consequently,

$$\sum_{b=1}^{n}\sum_{a=1}^{b}\sum_{c=1}^{n-(b-a)}\frac{N(a,b,c)p^{2n-(2b+c-a)}}{\mathbf{E}(X)}$$
$$\le \sum_{b=1}^{n}\sum_{a=1}^{b}\left(\frac{e}{\alpha}\right)^b\binom{b-1}{a-1}(2\gamma ep)^a\sum_{c=1}^{n-(b-a)}\frac{\gamma^c(n-b-a-2c)!}{c!p^c(n-b-a-c)!} \quad (5)$$

Now if $c \ge n^{2/3}$, then $c! \ge (c/e)^c$ and so

$$\sum_{c=n^{2/3}}^{n-(b-a)}\frac{\gamma^c(n-b-a-2c)!}{c!p^c(n-b-a-c)!} \le \sum_{c=n^{2/3}}^{n-(b-a)}\left(\frac{e\gamma}{cp}\right)^c = o(1).$$

If $c \le n^{2/3}$ and $n-b-a \ge n^{3/4}$, then

$$\sum_{c=1}^{n^{2/3}}\frac{\gamma^c(n-b-a-2c)!}{c!p^c(n-b-a-c)!} \le \left(\frac{2\gamma}{n^{3/4}p}\right)^c = o(1).$$

Finally, if $c \le n^{2/3}$ and $n-b-a \le n^{3/4}$, which implies that $b \ge (n-n^{3/4})/2$ since $b \ge a$, then we have

$$\sum_{b=(n-n^{3/4})/2}^{n}\sum_{a=1}^{b}\left(\frac{e}{\alpha}\right)^b\binom{b-1}{a-1}(2\gamma ep)^a\sum_{c=1}^{n^{2/3}}\frac{\gamma^c}{c!p^c} \le n^2\left(\frac{e}{\alpha}\right)^{(n-n^{3/4})/2}\left(\frac{\gamma}{p}\right)^{n^{2/3}} = o(1).$$

This proves that the R.H.S. of (5) is $o(1)$ and completes the proof of part (ii) of Theorem 1.

3 Final Remarks

Using the argument of McDiarmid [7] we obtain the same result for directed graphs.

It follows from the embedding theorem in Dudek, Frieze, Ruciński and Šileikis [3] that w.h.p. the random r-regular $G_{n,r}$ contains the square of a Hamilton cycle as long as $\sqrt{n}\log n \ll r \ll n$.

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