ON THE COMPLEXITY OF COMPUTING THE VOLUME OF A POLYHEDRON

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Abstract. We show that computing the volume of a polyhedron given either as a list of facets or as a list of vertices is as hard as computing the permanent of a matrix.

Key words. volume, polyhedra, complexity, \#P-complete

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1. Introduction. Recently there has been some interest in establishing the computational complexity of determining the volume of convex bodies in \( \mathbb{R}^d \). Elekes [2] and Bárány and Füredi [3] have shown that it is even difficult to closely approximate volumes of convex bodies defined by certain types of oracles. These hardness results complement the approximation algorithm of Lovász [4].

Lovász [4] also enquires about the complexity of computing the volume of a rational polytope given either by listing its facets or by listing its vertices. He conjectures that these problems are hard. In this paper we confirm Lovász's conjecture, and show that both problems are as hard as computing the matrix permanent, (see Valiant [6]).

We cannot quite describe the problems as \#P-complete, since they are not in the class \#P as defined by Valiant [6]. However, since we have no wish to define yet another complexity class, we state our results relative to \#P.

Let us now be more specific. Consider first Problem 1.

**Problem 1.** Let \( P = P(A, b) = \{ x \in \mathbb{R}^d : Ax \leq b \} \) be a polyhedron. \( A, b \) have rational entries where \( A = (a_i), i=1,2,\ldots,m \), \( j=1,2,\ldots,n \), and \( b = (b_i), i=1,2,\ldots,m \). We shall use the notation of Schrijver [5] to describe problem size. Thus if \( x = p/q \) is rational where \( p, q > 0 \) are relatively prime integers, then

\[
\text{size}(x) = 1 + \left[ \log_2 \left( \frac{p+1}{p} \right) \right] + \left[ \log_2 \left( \frac{q+1}{q} \right) \right]
\]

and

\[
\text{size}(A, b) = m(n+1) + \sum_{i=1}^{m} \text{size}(b_i) + \sum_{j=1}^{n} \sum_{i=1}^{m} \text{size}(a_{ij}).
\]

\( L = \text{size}(A, b) \) will be used as our measure of problem size. Since we can (by linear programming) determine in polynomial time whether or not \( \text{vol}(P) = 0 \) and whether or not \( \text{vol}(P) = \infty \), we can assume without loss of generality that \( 0 < \text{vol}(P) < \infty \).

We shall prove two theorems. The first shows that computing \( \text{vol}(P) \) is \#P-hard.

**Theorem 1.** Computing \( \text{vol}(P(A, b)) \) is \#P-hard, even when \( A \) is totally unimodular.

There is a technical difficulty in stating the converse of this theorem. It may be encapsulated in the following problem.

**Problem.** Is \( \text{size}(\text{vol}(P)) \) polynomially bounded in \( L \) ?

It is not too difficult to show that \( \text{vol}(P) \) is a rational, \( p/q \), say. We can also show that \( \text{vol}(P) \) cannot be too large. The difficulty is that \( q \) does not appear to have a
bound polynomial in $L$. The problem stated above thus seems to be an open (and possibly difficult) question. We show later that the difficulty disappears if we restrict ourselves to any class of polyhedra whose vertices can be scaled to become integer lattice-points by a polynomial size transformation. We conjecture that the answer to our problem is in the affirmative, but we are at present only able to state the following result in terms of approximation.

**Theorem 2.** Let $\epsilon > 0$ be rational. Given a $\#P$-complete oracle, in time polynomial in $L$ and size ($\epsilon$), we can compute $\hat{V}$ such that

$$|\hat{V} - \text{vol}(P(A, b))| < \epsilon.$$

**Corollary 1.** Suppose we restrict our attention in Theorem 2 to any class of polytopes for which size (vol $(P)$) is bounded by a polynomial in $L$. Then, using a $\#P$-complete oracle, vol $(P)$ can be computed exactly in time polynomial in $L$. □

Let us now consider Problem 2.

**Problem 2.** Let $X = \{X_1, X_2, \cdots, X_n\}$ be a set of rational points in $\mathbb{R}^n$. Let $P(X)$ be the convex hull of $X$. We shall consider the problem of computing vol $(P(X))$ when $X$ is given as an $n \times m$ matrix $(x_{ij})$, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m$. □

**Theorem 3.** Computing vol $(P(X))$ is $\#P$-hard. □

This time we have no difficulty in stating the converse result because, as we will show, size (vol $(P(X))$) is polynomially bounded in size $(X)$.

**Theorem 4.** Computing vol $(P(X))$ is $\#P$-easy. □

2. $\#P$-hardness of Problem 1. Let $B = \{0, 1\}$ and $C = B^n$. We will consider a single linear inequality

$$a^T x \leq b$$

in $\mathbb{R}^n$, with $a > 0$ an integer vector. Define the polytope

$$P = \{x \in \mathbb{R}^n : a^T x \leq b, 0 \leq x \leq e\}$$

where $e^T = (1, 1, \cdots, 1)$, and let

$$K = C \cap P.$$

We shall wish to regard (2.1) as being parameterised by $b$, in which case we will write $P(b)$ or $K(b)$ for emphasis. The following problem is known to be $\#P$-hard (see [3]).

**#KnapSack**

**Input.** Positive integers $a_1, a_2, \cdots, a_n, b$.

**Problem.** Determine $N = |K|$.

We Turing reduce $\#KnapSack$ to computing the volumes of certain polyhedra. For $x \in C$, let $|x| = e^T x$ denote the number of 1's in $x$.

Now consider the following problem, which may appear rather artificial, but will be required later.

**#Parity**

**Input.** As for $\#KnapSack$.

**Problem.** For $i = 0, 1$ let $N_i = |\{x \in K : |x| \equiv i \pmod{2}\}|$. Determine

$$D = N_0 - N_1.$$

**Lemma 1.** $\#Parity$ is $\#P$-hard.

**Proof.** Suppose we have a procedure for $\#Parity$. Let $M = e^T a + 1$, and consider the set of $(n + 1) \#Parity$ problems, as above, corresponding to the inequalities

$$(a + Me)^T x \leq b + rM \quad (r = 0, 1, \cdots, n).$$
Let \( N^{(r)}_r = \{ x \in K : |x| = r \} \). It follows, by an easy analysis, that the \( r \)-th \#PARITY problem defined by (2.5) determines the value
\[
D^{(r)} = (-1)^r N^{(r)} + \sum_{i=0}^{r-1} (-1)^i \binom{r}{i}.
\]

A straightforward calculation now yields
\[
N = 2^{n-1} + \sum_{r=0}^{n} (-1)^r D^{(r)}.
\]

Thus a polynomial number of calls to the \#PARITY procedure, plus polynomial additional time, would solve \#KNAPSACK.

We now turn to the proof of Theorem 1.

**Proof of Theorem 1.** Assume we have some procedure for determining \( \text{vol}(P) \).

Let \( \Delta = \{ x \in \mathbb{R}^n : a^T x \leq b, x \geq 0 \} \). Note that \( \Delta \) is bounded with \( \text{vol}(\Delta) = b^n / (n! \prod_{i=1}^n a_i) \).

Let \( U \) be the uniform measure on subsets of \( \Delta \) such that \( U(\Delta) = b^n \). Note that \( U \) is related to volume for a subset \( E \) of \( \Delta \) by the equation \( \text{vol}(E)/\text{vol}(\Delta) = U(E)/U(\Delta) \).

Thus our procedure for determining \( \text{vol}(P) \) can easily be modified to determine \( U(P) \), and we will assume this has been done. Let \( E_j = \{ x \in \Delta : x_j = 1 \} \) for \( j = 1, 2, \ldots, n \) and \( \bar{E} = (\Delta - E_1) \). Then clearly \( P = \bigcap_{j=1}^n E_j \). Now, for each \( v \in C \), define
\[
\bar{E}_v = \{ x \in \Delta : x > v \} = \bigcap_{v_i = 1} \bar{E}_i.
\]

Note that \( \bar{E}_v \) is either empty or is an \( n \)-simplex (actually its closure is). The well-known inclusion-exclusion formula (see, e.g., [7]) now yields
\[
U(P) = \sum_{v \in C} (-1)^r U(\bar{E}_v),
\]

It is easy to show that
\[
U(\bar{E}_v) = 0 \quad \text{if } a^Tv > b
\]
\[= (b - a^Tv)^n \quad \text{if } a^Tv \leq b.
\]

Thus
\[
U(P) = \sum_{v \in K(b)} (-1)^r (b - a^Tv)^n.
\]

Suppose that \( b \) is an integer, and \( \beta \) real such that \( b \leq \beta < b + 1 \). The integrality of \( a \) then implies that \( K(\beta) = K(b) \). Thus, writing \( F(\beta) = U(P(\beta)) \), it follows that
\[
F(\beta) = \sum_{v \in K(b)} (-1)^r (\beta - a^Tv)^n.
\]

Let \( \beta \) be a rational \( p/q \) with \( p, q > 0 \) relatively prime. It follows from (2.6) that \( q^n F(\beta) \) is an integer such that \( q^n F(\beta) \leq (2p)^n \). Thus size \( (F(\beta)) \leq n \text{size } (\beta) + 1 \). Thus the procedure for calculating \( U(P) \) can be used to determine \( F(\beta) \), provided \( \beta \) has size polynomial in \( L \). Now, expanding the terms in (2.6), we see that
\[
F(\beta) = \sum_{i=0}^{n} a_i \beta^{n-i}
\]
where
\[
a_i = (-1)^i \binom{r}{i} \sum_{v \in K(b)} (-1)^i (a^Tv)^i \quad (i = 0, 1, \ldots, n).
\]

Observe, in particular, that \( a_0 = \sum_{v \in K(b)} (-1)^v = D \), as defined in (2.4). This is \( P \)-hard to compute (Lemma 1). It follows, a fortiori, that it is \( P \)-hard to determine the coefficients of the polynomial \( F \). But we may do this as follows:
(i) Determine \( F(\beta) = U(P(\beta)) = n! \prod_{\alpha_j \in a_j \text{ vol } P(\beta)} \) for \( \beta = b + k/(n + 1), k = 0, 1, \ldots, n \).

(ii) Solve (in polynomial time) the resulting system of linear equations determined by \((2.7)\).

We observe that none of the numbers involved in the computations is very large. For example, it follows from \((2.8)\) that the \( \alpha_j \) are integers such that \( |\alpha| \leq (4e^T a)^n \).

There is one final point. The statement of Theorem 1 claims that it remains true when \( A \) is totally unimodular. This follows by making the substitution

\[ y_j = a_j, (j = 1, 2, \ldots, n) \]

in the system \((2.2)\) defining \( P \). The constraints are then easily transformed into a totally unimodular system. The theorem now follows. \( \square \)

Remark. The problem we have proved hard may be viewed in either of the following ways:

(i) If \( X \) is a point chosen randomly from the uniform probability distribution on the unit hypercube, then it is hard to compute the probability that \( X \) satisfies a single linear inequality.

(ii) Integrating the step-function

\[ \psi(x) = \begin{cases} 1 & (b - a^T x \leq 0) \\ 0 & (b - a^T x > 0) \end{cases} \]

over the unit hypercube is \( \#P \)-hard.

Note that the function \( \psi \) in (ii) is not even continuous. However, we may integrate \( \psi \) explicitly over \( k \) variables, using \((2.6)\). This gives the piecewise polynomial function

\[ \psi_k(x) = \sum_{\mathcal{K}(b - a^T x)} (-1)^k (b - a^T x - \hat{a}^T \hat{\psi})^k \]

where \( \hat{a}, \hat{\psi} \in \mathbb{R}^k \) and \( a, x \in \mathbb{R}^{n-k} \). This function is readily shown to be of class \( C^{k-1} \).

Hence we have the conclusion that it is hard to integrate such a function over the unit hypercube. In fact, since the description of \( \psi_k \) continues to be of polynomial size for \( k = O(\log n) \), the conclusion can be strengthened further.

3. \( \#P \)-easiness of Problem 1. We will go straight into the proof of Theorem 2.

Proof of Theorem 2. Since \( P \) is bounded, we have \([5, \text{Thm. 10.2}]\)

\[
(3.1) \quad P \subseteq \Gamma = \{ x \in \mathbb{R}^n : ||x|| \leq D = 2^{4e^T} (1, 2, \ldots, n) \}.
\]

Now let \( s = [mn^{2}2^{4e^T}L^{-n}/\varepsilon] \), and divide \( \Gamma \) into \( s^n \) subcubes of side \( \delta = 2D/s \). These subcubes fall into three classes:

(i) interior (wholly) to \( P \);
(ii) exterior (wholly) to \( P \);
(iii) boundary.

We will estimate \( V = \text{vol } P \) by the total volume of the interior cubes, \( \hat{V} \). Thus \( \hat{V} = I\delta^n \)

where \( I \) is the number of interior cubes. Clearly \( V \geq \hat{V} \), and the error \( E = V - \hat{V} \) is bounded above by \( J\delta^n \), where \( J \) is the number of boundary cubes. We show later that

\[
(3.2) \quad J \leq mn^2s^{n-1},
\]

and hence \( J\delta^n \leq \varepsilon \) as required.

The counting machine \( \delta \) \([6]\) which computes \( I \) works as follows. Each copy chooses one of the integer vectors \( t \), where \( 1 \leq t_j \leq s \) for \( j = 1, 2, \ldots, n \) in time proportional to \( s \) size \( s \). It then tests, in polynomial time, whether the subcube

\[
\Gamma' = \{ x \in \mathbb{R}^n : -D + (t_j - 1)\delta \leq x_j \leq -D + t_j\delta, j = 1, 2, \ldots, n \}
\]
is interior. Note that this is equivalent to

$$\frac{1}{2} \sum_{i=1}^{n} |a_i| \leq b_i + \sum_{i=1}^{n} a_i \delta \leq (i,\cdots,m).$$

It is clear that the sizes of all numbers involved, in particular $s$, are polynomial in $L$. The output of $\phi$ is $I$, from which $V$ is calculated.

It remains only to prove (3.2). If

$$P = \{ x \in \mathbb{R}^n : a_i^T x \leq b_i (i = 1, 2, \cdots, m) \}$$

then let $H_i = \{ x \in \mathbb{R}^n : a_i^T x = b_i \}$. Call a subcube $\Gamma'$ of $\Gamma$ intersected if $H_i \cap \Gamma' \neq \emptyset$ for some $1 \leq i \leq m$. Let $K$ be the number of intersected subcubes. Clearly $K \geq J$. Each subcube $\Gamma''$ projects onto $n$ squares $\Gamma_j'$ in the coordinate planes, i.e., if $\Gamma'' = \{ x \in \mathbb{R}^n : u_k \leq x_k \leq u_k + \delta \ (k = 1, 2, \cdots, n) \}$ then $\Gamma_j' = \{ x : x_j = 0 \text{ and } u_k \leq x_k \leq u_k + \delta \ (k \neq j) \}$. The total number of distinct squares is clearly $ns^{n-1}$. Construct a mapping from the intersected cubes to the squares as follows.

For each intersected cube $\Gamma'$ choose some $i$ such that $H_i \cap \Gamma' \neq \emptyset$ (e.g., the least such $i$). Choose $j$ so that

$$|a_i| = \max_{1 \leq k \leq n} |a_k|.$$

Suppose two intersected cubes map to the same square with the same value of $i$. Then there exist points $x, x'$ within the two cubes such that

$$a_i^T x = a_i^T x' = b_i,$$

$$x_j \leq x_j' \text{ and } |x_k - x_k'| \leq \delta \ (k \neq j).$$

Thus $x_j - x_j' \leq \sum_k a_k \delta / |a_i| \leq (n - 1) \delta$. Therefore at most $n$ cubes can map onto the same square for a given $i$. Since there are only $m$ values of $i$, at most $mn$ cubes map onto the same square. Hence $K = mn \times ns^{n-1} = mn \delta s^{n-1}$, and (3.2) follows.

Proof of Corollary 1. Suppose we know that size $(\text{vol} \ (P)) \leq p(L)$ for some polynomial $p(L)$. Take $\epsilon = 1/3p(L)^{1/2}$ in Theorem 2. We know that $V = a/b$ for integers $a, b \leq 2^{nL^2}$, and $V - \hat{V} \leq \epsilon$. Having computed $V$, we can use continued fractions to compute $\hat{V}$ exactly in polynomial time (see, e.g., [5, Cor. 6.3a]).

Finally, we give a simple example of a class of polyhedra for which $\text{vol} \ (P)$ has polynomial size.

Lemma 2. If $P$ is integral, then size $(\text{vol} \ (P)) = O(L^3)$.

Proof. Let $P = \{ x \in \mathbb{R}^n : a_i^T x \leq b_i (i = 1, 2, \cdots, m) \}$ and be such that every vertex of $P$ has integer coordinates. Now $P$ has a triangulation using only its own vertices. If $\sigma$ is a simplex of this triangulation, then

$$\text{vol} \ (\sigma) = \frac{1}{n!} \det \begin{pmatrix} \mathbf{1} & \mathbf{1} & \cdots & \mathbf{1} \end{pmatrix} = \frac{v}{n!}, \text{ say},$$

where $v, (i = 0, 1, \cdots, n)$ are vertices of $P$. Now $v_n$ is an integer. Hence $\text{vol} \ (P) = v/n!$, with $v = \sum v_n$, i.e.; $n! \text{vol} \ (P)$ is an integer. Now (3.1) implies

$$\text{vol} \ (P) \leq 2^n s^{nL^2} \cdot$$

Hence

$$\text{size (vol} \ (P)) \leq 8n^2 L + n + \lceil \log_2 (n! + 1) \rceil + 1,$$

and the result follows. □
4. \*P-hardness of Problem 2. We shall again reduce the counting problem
\#KNAPSACK to volume computations of the relevant form. We will make the
following two additional assumptions for \#KNAPSACK:

\[ (4.1) \]
There are no 0-1 solutions to \( a^T x = b. \)

We can ensure this simply by replacing \( a_j, j = 1, 2, \cdots, n \) and \( b \) by \( 2b + 1 \) in
(2.1). This does not affect the value of \( N. \)

\[ (4.2) \]
\( b > \frac{1}{2} e^T a. \)

If this is not true on input, then we can add a variable \( x_{n+1} \) with \( a_{n+1} = e^T a - b + 1, \)
and replace \( b \) by \( b + a_{n+1}. \) This adds \( 2^n \) to the value of \( N. \)

Now let \( P_1 = P_1 = \{ x \in \mathbb{R}^n : a^T x \leq b, 0 \leq x \leq e \}. \) Then (4.2) implies that \( \frac{1}{2} e = (\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}) \) is an interior point of \( P_1. \) Now, substituting \( y = 2x - e, \) transform \( P_1 \) to

\[ P_2 = \{ y \in \mathbb{R}^n : a^T y \leq b', -e \leq y \leq e \}, \]

where \( b' = 2b - e^T a > 0. \) Note that \( x \in [0, 1]^n \cap P_1 \iff y \in [-1, 1]^n \cap P_2. \) So we have
reduced \#KNAPSACK to computing \( N = |Y|, \) where

\[ Y = \{ y \in P_2 : y_j = \pm 1 (j = 1, 2, \cdots, n) \} = \{ y^{(1)}, y^{(2)}, \cdots, y^{(N)} \}, \]
say.

The constraints defining \( P_2 \) can be written as

\[ (4.3) \quad A^T y \leq 1, \quad -y_j \leq 1, \quad y_j \leq 1 \quad (j = 1, 2, \cdots, n) \]

where \( A = a / b'. \) Now \( 0 \) is an interior point of \( P_2. \) Consider the polar \( P_2 = [5, \text{Chap. 9}], \)
where

\[ (4.4) \]
\( P_2^* = \{ z \in \mathbb{R}^n : z^T u \leq 1 \text{ for all } u \in P_2 \} \)
\[ = \text{conv} \{ A, e_1, e_2, \cdots, e_n, -e_1, -e_2, \cdots, -e_n \}. \]

The second equality in (4.4) merely states the well-known relationship between the
facets of \( P_2 \) and the vertices of \( P_2^*. \)

We show that \( N \) can be computed from \( \text{vol} (P_2^*) \) and \( \text{vol} (\hat{P}_2^*), \) where \( \hat{P}_2^* \) is the
polyhedron obtained by using \( (b + \frac{1}{2}) \) in place of \( b \) in the definition of \( P_1 \) before
transforming to \( P_2^*. \) Since \( P_2^*, \hat{P}_2^* \) are defined as convex hulls, Theorem 3 will follow.

Now polarity yields a one-to-one correspondence between the (nondegenerate)
vertices of \( P_2 \) not lying in facet \( \{ y \in P_2 : A^T y = 1 \} \) (i.e., members of \( Y \)) and the (simplicial)
facets of \( P_2^* \) not containing the vertex \( A. \) Observe that the facet corresponding to \( y^{(i)} \)
has vertex set \( \Lambda^{(i)} = \{ y^{(i)}_1, e_j : j = 1, 2, \cdots, n \} \) for \( i = 1, 2, \cdots, N. \) Thus there is a decomposition
of \( P_2^* \) into simplices \( \sigma_1, \sigma_2, \cdots, \sigma_N, \) where \( \sigma_i = \text{conv} (\{A \cup \Lambda^{(i)} \}). \) Hence

\[ \text{vol} (P_2^*) = \sum_{i=1}^N \text{vol} (\sigma_i) \]
\[ = \frac{1}{n!} \sum_{i=1}^N \left| \det \begin{pmatrix} 1 & \cdots & 1 \\ A y^{(i)}_1 e_1 & \cdots & y^{(i)}_n e_n \end{pmatrix} \right| \]
\[ = \frac{1}{n!} \sum_{i=1}^N |1 - A^T y^{(i)}| \quad \text{since } y^{(i)}_j = \pm 1 \]
\[ = \frac{1}{n!} \sum_{i=1}^N \left| \frac{b - a^T x^{(i)}}{2b - e^T a} \right| \]
where $x^{(i)} = \frac{1}{2}(e + y^{(i)})$ is the zero-one solution to (2.1) which corresponds to $y^{(i)}$. Thus, using (4.2),

$$\frac{1}{2} n! \text{vol} (P_G) = \frac{b}{\sqrt{2}} \text{vol} (\hat{P}_G) = \frac{-N - b}{2b} = \frac{N - S}{2b} \frac{1}{a}$$

where $S = \sum_i a^T x^{(i)}$. Furthermore

$$\frac{1}{2} n! \text{vol} (\hat{P}_G) = \frac{(b + \frac{1}{2})}{2 \sqrt{2b}} N - \frac{1}{2} \frac{1}{a}$$

where we have used, in the notation of § 2, $K(b) = K(b + \frac{1}{2})$, so that $N$ and $S$ are unchanged.

Now $N$ can be easily computed from $\text{vol} (P_G)$ and $\text{vol} (\hat{P}_G)$. Finally note that $S$ is an integer with $0 < S \leq 2^b$, so that the numbers involved are of polynomial size. Thus Theorem 3 has been proved.

5. *P*-easiness of Problem 2. Let us first show that size $(\text{vol} (P(X)))$ is polynomially bounded. Indeed we can prove quite straightforwardly that, for $m > n \geq 1$,

$$\tau = \text{size} (\text{vol} (P(X))) \leq 3mn^2L$$

where $L = \text{size} (X)$. To see this, let $\lambda$ be the least common multiple of the denominators of entries of $X$. Then size $(\lambda) \leq mnL$. Now $\lambda X$ contains integer vectors and $\text{vol} (P(\lambda X)) = \lambda^n \text{vol} (P(X))$. Since $P(\lambda X)$ can be decomposed into simplices (see proof of Lemma 2), it follows that $n! \text{vol} (P(\lambda X))$ is an integer. Hence

$$\text{vol} (P(X)) = \frac{n! \lambda^n \text{vol} (P(X))}{n! \lambda^n}$$

expresses $\text{vol} (P)$ as a ratio of two integers. Noting that $\text{vol} (P(X)) \leq 2^{mL}$, (5.1) follows.

The proof of Theorem 4 is similar to that of Theorem 2, and so we will only give an outline. Now

$$P(X) \subseteq \Gamma_1 = \{ x \in \mathbb{R}^n : |x| \leq 2^k (j = 1, 2, \ldots, n) \}.$$

Let $s = [m^2 n \cdot 2^{6m^2 + n - 1} - 1]$ and divide $\Gamma_1$ into $s^n$ subcubes of size $\delta = 2^{-k} / s$. Our counting machine computes the number $I_i$ of subcubes which intersect $P(X)$. The estimate of volume is then $I_i \delta^n / m$. Note that this time we use an overestimate, rather than an underestimate, of the volume.

We can now reduce the testing for the intersection of an individual subcube with $P(X)$ to the solution of a single linear program.

To establish this claim, for simplicity let us assume we translate and scale the problem to that of testing the intersection of the cube $\Sigma = \{ x \in \mathbb{R}^n : -1 \leq x_j \leq 1 (j = 1, 2, \ldots, n) \}$, with the polytope $P = \text{conv} \{ x_1, x_2, \ldots, x_m \}$. Then $\Sigma \cap P = \emptyset$ if and only if there exists a $\gamma \in \mathbb{R}^n$ such that

$$\sum_{i=1}^m |\gamma_i| < 1 \quad \text{and} \quad \gamma^T x_i \geq 1 \quad (i = 1, 2, \ldots, m).$$

This can be tested by solving the linear program

minimize $z = \text{e}^T (\gamma_1 + \gamma_2)$

subject to $x_i^T (\gamma_1 + \gamma_2) \geq 1 \quad (i = 1, 2, \ldots, m)$

and by checking whether $\min z < 1$. 

\[\]
The error in the volume estimate is again bounded by the total volume of the subcubes which intersect the boundary of $P(X)$. Since $P(X)$ has less than $m^n$ facets, this is at most

$$m^n n^{1.5} s^{-1} \delta_i^n$$

(see the proof of Theorem 2 with $m$ replaced by $m^n$). By (5.1), this is at most $2^{-2(1.5-1)}$, and so $\text{vol}(P(X))$ can be computed exactly using continued fractions.

6. Remarks. Our results leave open two interesting questions. The first is the problem raised in §1 concerning the size of description of polyhedral volumes. The second is as to whether it remains hard to approximate the volume in either Problems 1 or 2; i.e., for some given $\epsilon > 0$, is it hard to obtain an estimate $\hat{V}$ of the volume $V$ such that $(1-\epsilon) V \leq \hat{V} \leq (1+\epsilon) V$? Our methods appear to shed little light on this issue, but we conjecture that this approximation problem is also hard. Finally, we observe that determining the volume of a polyhedron in fixed dimension is easy. We simply determine the complete face-lattice of the polyhedron, triangulate it, and then use the formula for the volume of a simplex.

REFERENCES


