Complexity of a 3-dimensional assignment problem

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We show that a certain 3-dimensional assignment problem is NP-complete. To do this we show that the following problem is NP-complete: given bipartite graphs $G_1, G_2$ with the same sets of vertices, do there exist perfect matchings $M_1, M_2$ of $G_1, G_2$ respectively such that $M_1 \cap M_2 = \emptyset$?

1. Introduction

Let $P, Q, R$ be 3 finite disjoint sets of equal size. For $u = (p, q, r) \in T = P \times Q \times R$ we define $s(u) = ((p, q), (p, r), (q, r))$ and for $A \subseteq T$ we let $s(A) = \bigcup_{u \in A} s(u)$.

A set $A \subseteq T$ is called a partial assignment if $u, v \in A$ implies $s(u) \cap s(v) = \emptyset$.

A total assignment $A$ is a partial assignment which satisfies $s(A) = (P \times Q) \cup (P \times R) \cup (Q \times R)$.

In this paper we prove the NP-completeness of the following 3-dimensional assignment problem (3DA):

Instance: disjoint finite sets $P, Q, R$ of equal size. A set $S \subseteq P \times Q \times R$.

Question: does there exist a total assignment $A \subseteq S$?

This is a special case of the integer programming problem (with $a_{ijk} = 0$ or 1)

$$\text{maximize } \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{m} a_{ijk} x_{ijk},$$

subject to

$$\sum_{i=1}^{m} x_{ijk} = 1, \quad j, k = 1, \ldots, m,$$

$$\sum_{j=1}^{m} x_{ijk} = 1, \quad i, k = 1, \ldots, m.$$
1, ..., m, (ii) no clause contains neglected variables.

**Question:** is there a truth assignment for \( V \) such that each clause in \( C \) has exactly one true variable?

**Theorem 2.1.** 1-3SAT \( \propto \) DM.

**Proof.** Suppose that \( C_i = \{ v(k) : k \in K_i \} \) where \( |K_i| = 3 \) for \( i = 1, \ldots, m \). We construct the following instance of DM:

\[
\begin{align*}
P &= X \cup D \cup F, \\
Q &= Y \cup E \cup G
\end{align*}
\]

where

\[
\begin{align*}
X &= \{ x[i, j] : i = 1, \ldots, 2m, j = 1, \ldots, n \}, \\
Y &= \{ y[i, j] : i = 1, \ldots, 2m, j = 1, \ldots, n \}, \\
D &= \{ d[i] : i = 1, \ldots, m \}, \\
E &= \{ e[i] : i = 1, \ldots, m \}, \\
F &= \{ f[i, k] : i = 1, \ldots, m, k \in K_i \}, \\
G &= \{ g[i, k] : i = 1, \ldots, m, k \in K_i \}.
\end{align*}
\]

**Definition of \( A_1 \).** For \( j = 1, \ldots, n \) let

\[
\begin{align*}
V[j] &= \{ (x[i, j], y[i, j]) : i = 1, \ldots, 2m \} \\
\overline{V}[j] &= \{ (x[i, j], y[i + 1, j]) : i = 1, \ldots, 2m \} \\
(\overline{y}[2m + 1, j] &= y[1, j]) \text{ then}
\end{align*}
\]

\[
A_1 = \bigcup_{j=1}^{n} (V[j] \cup \overline{V}[j]) \\
\cup \{ (d[i], e[i]) : i = 1, \ldots, m \} \\
\cup \{ (f[i, k], g[i, k]) : i = 1, \ldots, m, k \in K_i \}.
\]

We note (see Fig. 1) that if \( M \subseteq A_1 \) is a matching then for \( j = 1, \ldots, n \) we have

\[
\begin{align*}
M \supseteq V[j] \text{ and } M \cap \overline{V}[j] &= \emptyset \\
\text{(models } v(j) = \text{ true)}
\end{align*}
\]

or

\[
\begin{align*}
M \cap C[j] &= \emptyset \text{ and } M \supseteq \overline{V}[j] \\
\text{(models } v(j) = \text{ false}).
\end{align*}
\]

**Definition of \( A_2 \).** For \( i = 1, \ldots, m \) let

\[
\begin{align*}
\text{CS}[i] &= \bigcup_{k \in K_i} \{ (x[2i-1, k], y[2i-1, k]), \\
&(x[2i-1, k], y[2i, k]), \\
&(f[i, k], y[2i, k]), \\
&(f[i, k], e[i]), (d[i], y[2i-1, k]), \\
&(x[2i, k], g[i, k]) \}
\end{align*}
\]

(see Fig. 2).

For a given \( j \) let

\[
\{ x[i,j] : j \notin K_i \text{ where } t = [i/2] \} = \{ x[i(r), j] : r = 1, \ldots, s_j \}
\]

(defines \( i(1), i(2), \ldots \) where \( i(1) < i(2) < \cdots < i(s_j) \) and let

\[
Z[j] = \{ (x[i(r+1), j], y[i(r), j]) : r = 1, \ldots, s_j \}
\]

where \( i(s_j + 1) = i(1) \).

Note that \( Z[j] \cap (V[j] \cup \overline{V}[j]) = \emptyset \) provided that

**Assumption.** No variable \( v(j) \) occurs in exactly \( m - 1 \) clauses.

We can make the above assumption because there is a polynomial time algorithm that solves all instances of 1-3SAT that do not satisfy the as-

![Fig. 1. A_4 (m = 4; continuous edges are in V[j]; broken edges are in \( \overline{V}[j] \).](image-url)
Fig. 2. CS[i] (\(K_I = \{p, q, r\}\)).

**umption.

Now let \(Z = \bigcup_{i=1}^n Z[j]\) and let \(A_2 = \bigcup_{i=1}^m CS[i] \cup Z\). Note that \(Z \cap A_1 = \emptyset\).

For \(i = 1, \ldots, m\) and \(k \in K\), define (see Fig 3)

\[
CM[i, k] = \bigcup_{j \in K}\{(x[i, j], g[i, j])\}
\]

\[
\bigcup \{(d[i], y[2i - 1, k]), (x[2i - 1, k], y[2i, k]), (f[i, k], e[i])\}
\]

\[
\bigcup \bigcup_{j \in K - \{k\}} (x[2i - 1, j], y[2i - 1, j]), (f[i, j], y[2i, j])\}
\]

We next note that

if \(M \subseteq A_2\) is a matching then for \(i = 1, \ldots, m\) there exists \(k_i \in K\) such that \(M \cap CS[i] = CM[i, k_i]\) (2.2)

Note also the following properties of CM[i, k]:

\[
CM[i, k] \cap V[k] = \emptyset, \tag{2.3a}
\]

\[
CM[i, k] \cap \overline{V}[k] \neq \emptyset, \tag{2.3b}
\]

\[
CM[i, k] \cap V[j] \neq \emptyset, j \in K - \{k\}, \tag{2.3c}
\]

\[
CM[i, k] \cap \overline{V}[j] = \emptyset, j \in K - \{k\}, \tag{2.3d}
\]

\[
CM[i, k] \cap (V[j] \cup \overline{V}[j]) = \emptyset, j \notin K, \tag{2.3e}
\]

We must now show that 1-3SAT has a solution if and only if the above example of DM has a solution.

Suppose first that 1-3SAT has a satisfying assignment of truth values. In one such assignment let

\(T = \{j: v(j) = \text{true}\}\) and \(\overline{T} = \{1, \ldots, n\} - T\).

For \(i = 1, \ldots, m\) we can by assumption define \(k_i\), by \(T \cap K_i = \{k_i\}\). Then let

\[
M_1 = \bigcup_{j \in T} V[j] \cup \bigcup_{j \in \overline{T}} \overline{V}[j]
\]

\[
\bigcup \{(d[i], e[i]): i = 1, \ldots, m\}
\]

\[
\bigcup \{(f[i, k], g[i, k]): i = 1, \ldots, m, k \in K_i\}
\]

and

\[
M_2 = \bigcup_{i=1}^m CM[i, k_i] \cup Z.
\]

That \(M_1 \cap M_2 = \emptyset\) follows from (2.3) and \(Z \cap A_1 = \emptyset\).

Conversely suppose we are given disjoint matchings \(M_i \subseteq A_i\) for \(i = 1, 2\). Let \(T = \{j: V[j] \subseteq M_1\}\) and assign \(v(j) = \text{true}\) for \(j \in T\) and \(v(j) = \text{false}\) for \(j \in \overline{T}\).

Next (using (2.2)) let \(k_i\) be defined by \(M_2 \cap

Fig. 3. CM[i, r] (\(K_I = \{p, q, r\}\)).
CS[i] = CM[i, k_i] for i = 1, ..., m. It follows from (2.3b) that M_2 \cap \overline{V}[k_i] \neq \emptyset. Thus if M_1 \supseteq \overline{V}[k_i] we would have M_1 \cap M_2 \neq \emptyset. Thus M_1 \not\supseteq \overline{V}[k_i] and so by (2.1) we have M_1 \supseteq V[k_i] and k_i \in T for i = 1, ..., m. Now for j \in K_i - \{k_i\} we use (2.3c) and (2.1) in a similar manner to show that j \not\in T. Thus the truth value assignment satisfies all clauses in the required manner.

**Corollary 2.2.** 3DA is NP-complete.

**Proof.** We show that DM \preccurlyeq 3DA. Thus let p = \{p_1, ..., p_m\}, Q = \{q_1, ..., q_m\}, A_1, A_2 define an instance of DM. Let R = \{1, ..., m\} and let S = \cup_{i=1}^{m} S_i \subseteq P \times Q \times R where for i = 1, 2, S_i = A_i \times \{i\} and for i = 3, ..., m, S_i = P \times Q \times \{i\}.

We need only note that any total assignment \( A \subseteq S \) induces m disjoint matchings \( M_i = \{(p, q) : (p, q, i) \in A\} \) and that given a disjoint pair of matchings \( M_i \subseteq A, \) for \( i = 1, 2 \) we can easily extend them to a complete assignment. Indeed \( (P \times Q) - (M_1 \cup M_2) \) defines an \( m-2 \) regular bipartite graph which can be decomposed into \( m-2 \) disjoint matchings \( M_3, ..., M_m \). Then \( A = \cup_{i=1}^{m} (M_i \times \{i\}) \) forms a total assignment and \( A \subseteq S \).

We note next that 3DA is a special case of 3-dimensional matching (3DM).

**3DM**

**Instance:** disjoint finite sets \( X, Y, Z \) of equal size. A set \( T \subseteq X \times Y \times Z \).

**Question:** does there exist \( B \subseteq T \) such that each element of \( X \cup Y \cup Z \) occurs in exactly one member of \( B \)?

**Remark 2.3.** 3DA \( \preccurlyeq \) 3DM.

Given an instance \( P, Q, R, S \) of 3DA we proceed as follows: Let \( X = P \times Q, Y = P \times R, Z = Q \times R \) and \( T = \{((p, q), (p, r), (q, r)) : (p, q, r) \in S\} \). It is clear that \( S \) contains a total assignment if and only if \( T \) contains a matching.

We finally note another hard special case of 3DM that can be deduced from 3DA: there exist \( A \subseteq X \times Y, B \subseteq X \times Z, C \subseteq Y \times Z \) such that \( T = \{(x, y, z) : (x, y) \in A, (x, z) \in B \text{ and } (y, z) \in C\} \).

A practical instance of the above problem is described in Frieze and Yadegar [3].

This leads to an easy proof of NP-completeness of PARTITION INTO TRIANGLES (Garey and Johnson [4, pp. 68–69]) even when the graph under consideration is tripartite.

**3. Complexity of DM**

The instance of DM constructed in Theorem 2.1 has the following property: the bipartite graphs \( (P, Q, A_i) \) for \( i = 1, 2 \) are both planar and no vertex has degree exceeding 3.

If we restrict the instance of DM to those with vertex degrees bounded by 2 then the problem becomes polynomially solvable even if we have to find disjoint matchings \( M_i \) of graphs \( (P, Q, A_i) \) for \( i = 1, ..., k \).

**References**