K-GREEDY ALGORITHMS
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ABSTRACT  K-Greedy algorithms are natural generalisations of the classic Greedy Algorithm.

In this paper we discuss their application to the problem of maximising a separable concave function over sets of non-negative integer vectors satisfying a simple closure property.

We describe necessary and sufficient conditions for these algorithms to be truly optimising and conduct a worst-case analysis of a particular case when they are not.

The results generalise well known results on matroids and polymatroids.

We also give some results on an interchange algorithm.

§1. Introduction

Let $E$ be a finite set and let $F$ be a non-empty family of subsets of $E$ satisfying: if $I \subseteq F$ and $J \subseteq I$ then $J \subseteq F$. The members of $F$ are called independent sets and $(E,F)$ is called an independence system.

Many optimisation problems can be posed as: given $w: E \to \mathbb{R}$ the set of real numbers

\[(1.1) \text{ Maximise } w(A) = \sum_{e \in A} w(e) \text{ subject to } A \subseteq F\]

The greedy algorithm for this problem is

\[(1.2a) \ G: = \emptyset \]
\[(1.2b) \text{ Order the elements of } E = \{e_1, \ldots, e_n\} \text{ so that } \]
\[w(e_1) \geq w(e_2) \geq \ldots \geq w(e_k) > 0 \geq w(e_{k+1}) \geq \ldots \geq w(e_n) \]
\[(1.2c) \text{ For } i=1, \ldots, k \text{ do if } G \cup \{e_i\} \subseteq F \text{ then } G := G \cup \{e_i\}\]
\[(1.2d) \text{ Output the greedy solution } G:\]

The performance of the greedy algorithm is well understood:

(PG$_1$) For a fixed independence system $(E,F)$ the greedy algorithm produces a maximum weight independent set for arbitrary $w$ if and only if $(E,F)$ is a matroid – see Edmonds [2] or Lawler [11] or Welsh [12].

(PG$_2$) If $(E,F)$ is not necessarily a matroid and for given $w$ $M$ is a maximum weight independent set then

\[(1.3) \ w(G) \geq q(E,F)w(M)\]

where $q(E,F)$ is the rank quotient (defined later on). Furthermore for any $(E,F)$ there exists $w$ such that (1.3) holds with equality – see Jenkyns [7] or Korte and Hausmann [9].

The purpose of this paper is to generalise the greedy algorithm and (i) describe those systems for which the generalisation optimises and (ii) provide some worst-case analysis when it does not.

In §2 we describe a greedy approach to a generalisation of problem (1.1). The characterisation of when this approach works generalises PG$_1$. In §3 we analyse the worst-case performance of a particular example of this method. This leads to a generalisation of PG$_2$. 
The results of this paper generalise previous results given in Frieze [4] and are related to results given in Hausmann, Jenkyns and Korte [6] and Jenkyns [8].

In §4 we discuss an interchange heuristic.
§2. A Greedy Algorithm for a Generalised Independence System

Let \( m, n \) be positive integers and let \( M = \{0, 1, \ldots, m\} \).

A Generalised Independence System, GIS, is a set \( X \subseteq M^n \) satisfying:
\[ x \in X \text{ and } x \geq y \in M^n \text{ implies } y \in M^n. \]
(Vectors are compared component-wise).

When \( m = 1 \) if we take \( E = \{1, \ldots, n\} \) and \( F = \{I(x) : x \in X\} \) where \( I(x) = \{j : x_j = 1\} \) then \((E, F)\) is an independence system and conversely.

GIS's can for example be the solution sets of integer programs with non-negative constraint matrices. Greedy algorithms and GIS's are discussed in Edmonds [3] and Dunstan and Welsh [1].

The optimisation problem discussed here is the following: given concave functions \( w_j : M \to \mathbb{R} \) with \( w_j(0) = 0 \) for \( j = 1, \ldots, n \)

(2.1) Maximise \( w(x) = \sum_{j=1}^{n} w_j(x_j) \) subject to \( x \in X \)

Note that when \( m = 1 \) (2.1) 'reduces' to (1.1).

Notation

For \( j = 1, \ldots, n \) \( u_j(w) = \max \{x \in M : w_j(x) > w_j(x-1)\} \)

\( w_j(-1) \) is defined to be \( -\infty \).

For \( x \in X \) \( |x| = \sum_{j=1}^{n} x_j \)

For \( x \in X \) \( \text{ADD} (x, w, k) = \{y \in X : |y| = k \text{ and } x + y \in X\} \)
and \( x_j + y_j \leq u_j(w) \) for \( j = 1, \ldots, n \).

The greedy algorithm to be described has a parameter \( K \). For each positive integer \( t \), \( K(t) \) is a set of non-negative integers satisfying
\( \phi \cap K(t) = \{0, 1, \ldots, t-1\} \).

K-Greedy Algorithm \( GA(K) \)

The algorithm computes a set of vectors \( \mathbf{g}(t) \) where \( |\mathbf{g}(t)| = t \) in a 'greedy' manner.
(S0) \( t:=0; g^{(0)}=0 \)
(S1) \( t:=t+1 \)
(S2) For \( k \in K(t) \) let \( S_k = \text{ADD}(g^{(k)}, w, t-k) \)
If \( S_k = \emptyset \) for all \( k \in K(t) \) go to (S3)
else for each \( S_k \neq \emptyset \) define \( \gamma^{(k)} \) by
\[
\gamma^{(k)} = \max(\gamma^{(k)}, \forall \gamma \in S_k)
\]
Then define \( g^{(t)} \) by
\[
w(g^{(t)}) = \max(w(g^{(t)} + \gamma^{(k)}), \forall k \in K(t))
\]
go to (S1)
(S3) Output \( g \) where \( w(g) = \max(w(g^{(s)}), s < t) \)

If \( m=1 \) and \( K(t) = \{t-1\} \) then we have the greedy algorithm of §1. If \( m \geq 1 \) and \( K \) is the same then we have the greedy algorithm of Dunstan and Welsh (more or less).

We can give a complete characterisation of those GIS's for which GA(K) is always valid, only when \(|K(t)| = 1\) for all \( t \). If \(|K(t)| \geq 2\) our characterisation is only sufficient. (\(|A|\), where \( A \) is a set, denotes its cardinality.)

Note the somewhat 'premature' stopping rule in (S2) – it may be possible to continue with larger \( t \). This rule is necessary to prove theorem 2.2 below.

A GIS \( X \) is said to be a \textbf{K-matroid} if the following holds:
(2.2) If \( z \in X \) and \( |z| = t \) and \( x^{(i)}_i \in K(t) \) satisfy \(|x^{(i)}_i| = i \) for \( i \in K(t) \) then there exist \( k \in K(t) \) and \( \gamma \) such that
(a) \(|\gamma| = t-k \) \quad (b) \( x^{(k)}_i + \gamma \in X \) \quad (c) \( y_j > 0 \) if \( j \leq z_j - x^{(k)}_j \)

Thus if \( m=1 \) and \( K(t) = \{t-1\} \) (2.2) is a 0-1 vector form of matroid axiom. If \( m \geq 1 \) and \( K(t) = \{t-1\} \) then K-matroids are the integer points of integer polymatroids.
Theorem 2.1

If $X$ is a $K$-matroid then $GA(K)$ solves (2.1).

Proof

We show by induction on $p$ that if $z \in X$ and $|z| = p$ and $z_j = u_j(w)$ for $j = 1, \ldots, n$ then $g^{(p)}$ is defined and $w(z) \leq w(g^{(p)})$. Since $w_j$ is concave for $j = 1, \ldots, n$ no better $z$ exists with $z_j > u_j(w)$.

The induction hypothesis is clearly true for $p = 0$ and so assume it to be true for $p < q$. Let $z \in X$ and $|z| = q$. The inductive assumption implies $g^{(q)}(t)$ exists for $t \in K(q)$. From (2.2) there exists $y$ and $k$ with properties (a) - (c). Note that (c) implies that

$$g_j(k) + y_j \leq \max(g_j(k), z_j) \leq u_j(w).$$

Thus $ADD(g^{(k)}, w, q-k) \neq \emptyset$ and so $g^{(q)}(q)$ exists. Now

$$w(g^{(q)}(q)) \geq w(g^{(k)} + y)$$

$$= w(g^{(k)}) + \sum_{j=1}^n (w_j(g_j(k)) + y_j) - w_j(g_j(k))$$

$$\geq w(z) + \sum_{j=1}^n (w_j(z_j) - w_j(z_j - y_j))$$

by induction and the concavity of $w_j$ for $j = 1, \ldots, n$

$$= w(z)$$

We can prove the following restricted converse:

Theorem 2.2

If $|K(t)| = 1$ for $t > 0$ and $GA(K)$ is guaranteed to solve 2.1 for arbitrary concave $w$ then $X$ is a $K$-matroid.

Proof

Let $K(t) = \{k(t)\}$ and $z, x \in X$ satisfy $|z| = p$ and $|x| = k(p)$. Define $w_j$ for $j = 1, \ldots, n$ by

$$w_j(\xi) = \xi \quad 0 \leq \xi \leq u_j = \max(x_j, z_j)$$

$$= u_j \quad u_j < \xi$$

Note that $w(y) \leq |y|$ for all $y \in X$. It follows that $GA(K)$ could at some stage produce $g^{(k(p))} = \xi$. If the algorithm were to stop before $t$ reached $p$ then $w(g) < w(z)$. Similarly on $t$ reaching $p$ we must have $S = ADD(g^{(k(p))}, w, p - k(p)) \neq \emptyset$. But for $y \in S$ $x_j + y_j \leq u_j(w) = u_j$ implies that $x_j + y_j \leq z_j$ whenever $y_j > 0$ for $j = 1, \ldots, n$ and so (2.2) holds.
Theorems 2.1 and 2.2 can be seen to generalise known results on matroids and polymatroids by taking $K(t) = \{t-1\}$.

It is not difficult to construct $K$-matroids for different $K$ that are not matroids or polymatroids.

For example consider the set $X$ of solutions to

$$p \sum_{i=1}^{r} x_i + (p+1) \sum_{i=r+1}^{n} x_i \leq ap + k$$

$$0 \leq x_i \leq m \text{ and integer}$$

where $a, p, k, r, n$ are positive integers and $m, m(n-r) \geq a > k+1$. This is not a $K$-matroid for $K(t) = \{t-1\}$ as can be seen by choosing $x^{(0)}, x^{(1)} \in X$ with $|x^{(0)}| = \sum_{i=1}^{r} x_i^{(0)} = a$ and with $|x^{(1)}| = \sum_{i=r+1}^{n} x_i^{(1)} = a-1$.

However $X$ is an $M_\kappa$-matroid where $M_\kappa(t) = \{\max(0, t-k)\}$. In this case $GA(M_\kappa)$ is the $k$-greedy algorithm considered in [4].

When $K(t) = k[t/k]$ we have an 'enhancement' of the $k$-greedy algorithm of [6].

Note that being a $M_\kappa$-matroid implies being an $M_{k\ell}$-matroid for $\ell \geq 1$ and so matroids and polymatroids which are $M_1$-matroids form subclasses of this type of system.

It has not however been possible to find interesting $K$-matroids for which $GA(K)$ provides a new efficient (polynomial time) algorithm other than of course for $K = M_1$.

This is rather disappointing but one lives in hope. One can however look at this from another viewpoint. For some functions $K$ e.g. $K(t) = \{t-1, \ldots, \max(t-k, 0)\}$ for a fixed $k$ the $K$-greedy algorithm is not at all an unnatural generalisation of the standard greedy algorithm. The preceding analysis indicates however that this is not likely to be an optimising algorithm for any interesting problem.
Finally in this section we note that theorem 2.1 is still true if we define $|x| = \sum_{j=1}^{n} a_j x_j$ where $a_1, \ldots, a_n$ are positive integers and $X$ satisfies the extra axiom (trivial for $a_1 = \ldots = a_n = 1$)

(2.3) $|x| = t$ and $k \in \mathbb{K}(t)$ implies there exists $y \leq x$ such that $|y| = k$.

The proof of theorem 2.1 is unchanged. The extra axiom is used to imply the existence of $\mathbb{g}(t)$ for $t \in \mathbb{K}(q)$.

As an example if $a_j \in \{1, 2\}$ for $j = 1, \ldots, n$ and

$X = \{x \in \mathbb{N}^n : \sum_{j=1}^{n} a_j x_j \leq L\}$ for some positive integer $L$ then $X$ is an $M_2$-matroid.

Finally for theorem 2.2 if we assume (2.3) we can show (2.2) assuming $GA(K)$ always solves problem 2.1. However we cannot deduce (2.3) as opposed to assuming it.
§3. Worst-Case Analysis

For the remainder of the paper we restrict our attention to independence systems defined in §1 and revert to sets as opposed to vectors. We shall assume \( E = \{1, 2, \ldots, n\} \) and in problem 1.1 use \( w(j) \) and \( w_j \) interchangeably.

Notation

For \( S \subseteq E \) a basis of \( S \) is a maximal (with respect to inclusion) independent subset of \( S \).

\[
\begin{align*}
\text{ur}(S) &= \max(|I| : I \text{ is a basis of } S) = \text{upper rank of } S \\
\text{lr}(S) &= \min(|I| : I \text{ is a basis of } S) = \text{lower rank of } S
\end{align*}
\]

The rank quotient \( q(E, F) \) is defined by

\[
q(E, F) = \min(S \subseteq E) \left( \frac{\text{lr}(S)}{\text{ur}(S)} \right)
\]

Inequality (1.3) can be interpreted as 'the worst-case performance of the greedy algorithm occurs when \( w_j = 1 \) or 0'.

Equivalently for this algorithm we can say \( w_j = 1 \) or \( w_j < 0 \). We shall generalise this statement to a wider class of greedy algorithm.

Specifically now let \( K \) be a sequence \( k_1, k_2, \ldots, k_a \) of positive integers satisfying \( k_1 + \ldots + k_a > n \).

Let \( w: E \to \mathbb{R} \) be given. The \( K \)-greedy algorithm of this section is

\[
\begin{align*}
(S0) & \quad G_K := \emptyset; \quad t := 0 \\
(S1) & \quad t := t + 1 \\
(S2) & \quad \text{Let } S = \{ I : |I| = k_t, I \cap G_K = \emptyset, I \cup G_K \subseteq F, w(I) > 0 \text{ and } w_j > 0 \text{ for } j \in I \}\} \\
& \quad \text{If } S = \emptyset \text{ go to (S3)} \\
& \quad \text{else let } w(I*) = \max(w(I) : I \in S) \\
& \quad G_K = G_K \cup I* \\
& \quad \text{go to (S1)} \\
(S3) & \quad \text{Output } G_K \text{ as solution.}
\end{align*}
\]
Notation

For a fixed \((E,F)\) and a given \(w\) and sequence \(J=j_1,j_2,\ldots,j_b\) where \(j_1+\ldots+j_b \geq n\) let
\[
F_J(w) = \{I \in \mathcal{F} : |I| \in \{j_1,j_1+j_2,\ldots,j_1+\ldots+j_b\} \text{ and } w_I \geq 0 \text{ for } i \in I\}
\]
If \(F_J(w) \neq \emptyset\) define \(M_J(w)\) by \(w(M_J(w)) = \max\{w(I) : I \in F_J(w)\}\)
otherwise \(M_J(w) = \emptyset\).

Also let \(F_d = \{I \in \mathcal{F} : |I| \geq d\}\) for non-negative integer \(d\).

Next let
\[
WORST(J,K) = \sup\{w(M_J(w))/w(G_K) : w \text{ satisfies } w(G_K) > 0 \text{ for } G_K \text{ chosen by K-greedy}\}
\]

For \(j_1 < k_1\) we can have \(WORST(J,K) = \infty\) if there is some \(I \in \mathcal{F}\) with \(|I| = j_1\) and an \(i \in I\) such that \(i \notin J\) for any \(J \in F_{k_1}\). One simply makes \(w_i\) large and \(w_j = 1\) for \(j \neq i\).

The worst-cast result we have proved requires:

\[
\begin{align*}
(3.1a) & \quad j_1 \geq k_1 \\
(3.1b) & \quad j_r < k_s \quad \text{for any } r \geq 1, s \geq 2.
\end{align*}
\]

The following lemma requires no proof:

**Lemma 3.1**

If \(|J| = t\) and \(|S| \geq t\) and \(w(J) \geq w(I)\) for all \(I \subseteq S\) with \(|I| = t\) then \(w(S)/|S| \leq w(J)/|J|\).

Furthermore the second and third inequalities can be reversed.

Keeping \(J,K\) fixed now abbreviate \(WORST(J,K)\) by \(WST\).

We ignore the moment the trivial case with \(F_{k_1} = \emptyset\) and \(WST\) undefined.

It follows from the lemma that \(WST \leq n/k_1\) if \(F_{k_1} \neq \emptyset\).

Elementary topological arguments can be used to show that there exist \(\hat{w},\hat{M} = M_J(\hat{w})\) and K-greedy solution \(\hat{G}\) such that \(WST = \hat{w}(\hat{M})/\hat{w}(\hat{G})\).

We assume that \(|\hat{M}| + |\hat{G}|\) is as small as possible under these circumstances.

Now let \(\hat{M} = \{m_1,\ldots,m_a\}\) where \(w(m_i) > w(m_{i+1})\) for \(1 \leq i < a\). If \(a = j_1 + \ldots + j_p\) we define \(M_t = \{m_{k_{t-1} + 1},\ldots,m_{k_t}\}\) where \(k_t = j_1 + \ldots + j_t\) for \(0 \leq t \leq p\) assuming \(k_p = a\) and then define \(X_i = \hat{w}(M_i)\) for \(1 \leq i \leq p\).
Next let \( \hat{G} = \bigcup_{i=1}^{q} G_i \) where \( G_1, \ldots, G_q \) are the \( q \) successive augmentations in step (S2) of the particular application that produced \( \hat{G} \). Let \( Y_i = \hat{w}(G_i) \) for \( 1 \leq i \leq q \).

The following properties of \( X_1, \ldots, X_p, Y_1, \ldots, Y_q \) follow from lemma 3. and the definition of K-greedy.

\[
\begin{align*}
(3.2) & \quad \frac{X_t}{j_t} = \frac{X_{t+1}}{j_{t+1}} \quad \text{for } 1 \leq t < p \\
(3.3) & \quad \frac{Y_t}{k_t} = \frac{Y_{t+1}}{k_{t+1}} \quad \text{for } 1 < t < q
\end{align*}
\]

We next define the function \( h \) by

\[
(3.4) \quad h(r) = \min(s : \frac{Y_r}{k_r} \geq \frac{X_s}{j_s})
\]

We show later that \( h(q) \leq p \) and it follows that

\[
(3.5) \quad 1 = h(1) \leq h(2) \leq \ldots \leq h(q) \leq p
\]

**Notation**

Let \( \square \) represent a symbol used to identify some entity which is indexed by the positive integers. We use the notation \( \square [i:j] \) to represent \( \square_i + \square_{i+1} + \ldots + \square_j \) e.g. \( X[i:j] = X_i + \ldots + X_j \)

Note that when \( j < i \) we take \( \square [i:j] = 0 \).

The following lemma is the basis for our result.

**Lemma 3.2**

(a) \( \frac{Y_s}{k_s} \geq \frac{X_r}{j_r} \) or \( \text{WST} \geq \frac{X[r+1:p]}{Y[s:q]} \) for \( 1 \leq r \leq p \) and \( 2 \leq s \leq q \)

(b) \( \frac{Y_q}{k_q} \geq \frac{X_p}{j_p} \)

**Proof**

(a) Let \( S = S \cup \bigcup_{i=1}^{r} G_i \) and \( T = S \cup \bigcup_{i=1}^{r} M_i \). If there exists \( Z \subset T \setminus S \) with

\[
|Z| = k_S \quad \text{and } S \cup Z \in F
\]

then we deduce from the definition of the algorithm that \( Y_s \geq \hat{w}(Z) \geq \hat{w}(A) \) where \( A = \{m_{i_r} - k_S, \ldots, m_{i_r} \} \). Lemma 3.1 plus the fact that \( k_S \geq j_r \) implies \( \hat{w}(A)/k_S \geq \frac{X_r}{j_r} \). - note 3.1b.

On the other hand if no such \( Z \) exists then we can define \( \tilde{w} \) by

\[
\tilde{w}_i = \hat{w}_i \quad \text{for } i \in T \quad \text{and } \tilde{w}_i = -1 \quad \text{otherwise.}
\]

Thus using \( \tilde{w} \) the algorithm could choose \( S \) as its solution. We deduce then that

\[
(3.6) \quad \text{WST} \geq X[1:r]/Y[1:s-1]
\]
The result follows as $\text{WST}=X[1:p]/Y[1:q]$

(b) If $q=1$ we can apply lemma 3.2 otherwise let $r=p$ and $s=q$ and argue as in (a) up to (3.6). Thus $Y_q/k_q \geq X_p/J_p$ or $\text{WST} \geq X[1:p]/Y[1:q-1]$. The latter inequality is impossible as $Y_q > 0$.

The next two lemmas are concerned with bounding the ratio of two sums of real numbers satisfying the given properties of the $X_i,Y_j$.

**Lemma 3.3**

Let real non negative numbers $x_1, \ldots, x_p, y_1, \ldots, y_q$ satisfy

(3.7a) \[ x_1/j_1 \geq x_2/j_2 \geq \ldots \geq x_p/j_p \]

(3.7b) \[ y_1/k_1 \geq y_2/k_2 \geq \ldots \geq y_q/k_q \]

(3.7c) \[ y_1 > 0 \]

(3.7d) \[ x_h(s)/j_h(s) \leq y_s/k_s \]

for positive integers $h(1), \ldots, h(q)$ satisfying

(3.7e) \[ 1 = h(1) \leq h(2) \leq \ldots \leq h(q) \leq p \]

If $\rho = X[1:p]/Y[1:q]$ then

(3.8) \[ \rho \leq \max(j[1:t]/k[1:g(t)]:1 \leq t \leq p) \]

where $g(t) = \max(i:h(i) \leq t)$

**Proof**

For fixed values of $x_1, \ldots, x_p$ $\rho$ will be maximised by putting $y_s = k_s x_h(s)/j_h(s)$ for $1 \leq s \leq q$. We can scale these quantities to satisfy $y[1:q]=1$.

Then defining $z_t = x_t/j_t$ for $1 \leq t \leq p$ we have

$\rho \leq \max j_1 z_1^+ \ldots j_p z_p$

subject to

(3.9) \[ k_1 z_h(1)^+ \ldots k_q z_h(q) = 1 \]

\[ z_1 \geq z_2 \geq \ldots \geq z_p \geq 0 \]

Equation (3.9) can be restated as

$k[1:g(1)]z_1 + k[g(1)+1:g(2)]z_2 + \ldots + k[g(p-1)+1:g(p)]z_p = 1$
Thus defining $u_p = z_p$ and $u_t = z_t - z_{t+1}$ for $1 \leq t < p$ we obtain

$$\rho \leq \max j_1 u_1 + j_{[1:2]} u_2 + \ldots + j_{[1:p]} u_p$$

subject to

$$k_{[1:g(1)]} u_1 + k_{[1:g(2)]} u_2 + \ldots + k_{[1:g(p)]} u_p = 1$$

$$u_1, \ldots, u_p > 0$$

The solution to the above linear program is well known and gives the required result.

**Lemma 3.4**

Suppose that the conditions of lemma 3.3 are satisfied and further that $h(t) = 1$ or

$$\rho \leq x[h(t):p]/y[t:q]$$

holds for $1 \leq t < q$. Then

$$\rho \leq j_{[1:p]}/k_{[1:q]}$$

**Proof**

We shall prove first using a backward induction that $h(t) > 1$ implies

$$\rho \leq j_{[h(t):p]}/k_{[t:q]}$$

If $h(q) > 1$ then (3.12) is obvious from (3.7) and (3.10). Suppose then there exists $s < q$ such that $h(s) > 1$ and that (3.12) holds for all $t > s$. Applying lemma 3.3 to RHS of (3.10) with $t = s$ gives

$$\rho \leq \max (j[h(s):u]/k[s:g(u)] : h(s) \leq u \leq p)$$

Let $u^*$ denote a value of $u$ maximising the RHS of (3.13). If $u^* = p$ then (3.13) is (3.12) with $t = s$.

So assume $u^* < p$ as $u^*$ maximises the RHS of (3.13) we must have $g(u^* + 1) > g(u^*)$ else $u^* + 1$ is better than $u^*$. But this implies

$$h(g(u^*) + 1) = u^* + 1$$

Thus

$$\rho \leq j_{[h(s):h(g(u^*) + 1) - 1]}/k[s:g(u^*)]$$

Combining (3.15) and (3.12) with $t = g(u^*) + 1 > s$ gives (3.12) with $t = s$ and completes the induction.
We now use this result in (3.8). Thus let \( t^* \) be a value of \( t \) maximising the RHS of (3.8). If \( t^* = p \) we have (3.11) and so assume \( 1 \leq t^* < p \). We argue as before that (3.14) holds this time with \( u^* \) replaced by \( t^* \). Thus

\[
\rho \leq j[1:h(g(t^*)+1)-1]/k[1:g(t^*)]
\]

Now \( h(g(t^*)+1) > 1 \) by (3.14) and so (3.12) holds with \( t = g(t^*)+1 \)

Combining this with (3.16) gives \( \rho \leq j[1:p]/k[1:q] \)

We have already shown that the \( X_i, Y_j \) and \( h \) as defined prior to lemma 3.2 satisfy the conditions of lemma 3.4. Our next task is simply to put this result in a more convenient form.

**Notation**

Let \( S \subseteq E \) and \( D = d_1, \ldots, d_s \) be a sequence of positive integers with \( d[1:s] \geq n \). Let \( S(D) = \{d_1, d[1:2], \ldots, d[1:s]\} \). Let \( F_D = \{I \in F : |I| \in S(D)\} \).

A \( D \)-basis of \( S \) is a set \( I \) satisfying

(1) \( I \subseteq S \)

(2) \( I \in F_D \)

(3) no \( J \) strictly containing \( I \) satisfies (1) and (2).

Let \( ur_D(S) = \max(|I| : I \text{ is a } D \text{-basis of } S) \)

\( lr_D(S) = \min(|I| : I \text{ is a } D \text{-basis of } S) \)

where \( ur_D(S) = 0 \) and \( lr_D(S) = 1 \) if \( S \) contains no \( D \) bases.

For the given sequences \( J, K \) define \( W(J, K) = \max(ur_J(S)/lr_K(S) : S \subseteq E) \)

**Theorem 3.1**

If \( J, K \) satisfy (3.1) then

\[
WST = \text{Worst}(J, K) = W(J, K)
\]

**Proof**

Equation (3.17) is trivial if \( F_J \) or \( F_K = \emptyset \) and so we assume \( F_J, F_K \neq \emptyset \)

\( a \) \( WST \geq W(J, K) \)

Let \( T, U, V \) be such that \( W(J, K) = ur_J(T)/lr_K(T) \) and \( ur_J(T) = |U| \) and \( lr_K(T) = |V| \) where \( U \) is a \( J \)-basis of \( T \) and \( V \) is a \( K \)-basis of \( T \). Define \( w \) by \( w(e) = 1 \) for \( e \in U \cup V \) and \( w(e) = -1 \) otherwise. Then \( M_J(w) = U \) and \( G_K \) could be \( V \) giving \( WST \geq |U|/|V| \).

\( b \) \( WST \leq W(J, K) \)
Let $\hat{G}, \hat{M}$ be as before and let $\hat{T} = \hat{M} \cup \hat{G}$. Now $X_p > 0$ else $|\hat{M}|$ can be made smaller. Thus $\hat{G}$ is a $K$-basis of $T$ for if there exists $Z \in \hat{M} - \hat{G}$ with $\hat{G} \cup Z \in F$ and $|Z| = k_{q+1}$ then $w(Z) > 0$ as $k_{q+1} > J_p$. But this implies the $K$-greedy algorithm stopped prematurely. Thus

$$W(J,K) > |\hat{M}| / |\hat{G}|$$

by lemma 3.4.

Note that $PG_2$ of §1 follows by taking $J = K = 1, 1, 1, \ldots 1$ does not hold then theorem 3.1 in our proof of lemma 3.2. If 3.1 does not hold then theorem 3.1 can fail to hold. Consider the following example:

$E = \{e_1, \ldots, e_5, f_1, \ldots, f_7\}$. The maximal elements of $F$ are

$I_e = \{e_1, \ldots, e_5\}, I_f = \{f_1, \ldots, f_7\}, \{e_i, e_j, e_k, f_6, f_7\}$,

$\{e_i, e_j, f_k, f_6, f_7\}, \{e_i, f_j, f_k, f_6, f_7\}$

where $1 \leq i, j, k \leq 5$

Let $J = 1, 3, 3, 3, \ldots$ and $K = 1, 2, 2, 2, \ldots$. One can check that

$W(J,K) = 7/5$ but putting $w(g) = 1$ for $g \in E - \{e_4, e_5, f_6, f_7\}$ and $w(g) = 1/2$ otherwise gives $w(I_f) = 6$ and $w(I_e) = 4$. Thus $WST \geq 6/4$ as $I_e$ is a possible $K$-greedy solution.

We next consider the case where $J$ and $K$ are of the form

$Z(i,k) = i, k, k, k, \ldots$ for some $i \leq k$. If $J = Z(i_1, k)$ and $K = Z(i_2, k)$ where $i_1 \geq i_2$ then theorem 3.1 applies. If we apply the $Z(i,k)$-greedy algorithm for $i = 1, \ldots, k$ and then take best of the $k$ solutions generated we can use theorem 3.2 to show that this will be valid for arbitrary $w$ if and only if $(E,F)$ is a $M_K$-matroid.

We can give no non-trivial worst case results for the greedy algorithm of §2. In the case of an independence system where $|K(t)| = 1$, although it can be split into a number of applications of the algorithm of this section the rules for termination are incompatible. If the termination rules are made to coincide the best we have been able to do is to apply theorem 3.1 separately. This is too weak even to determine if the bound $q(E,F)$ for the ordinary greedy algorithm is achieved.
§4. Interchange Algorithms

We consider here a nice property of $M_k$-matroids which generalises a known property of matroids.

**Theorem 4.1**

Let $(E,F)$ be an $M_k$-matroid. If $I* \in F$ is such that

\[(4.1) \quad w(I*) \geq w(I) \text{ for all } I \in F \text{ with } |I| = |I*| \text{ and } |I-I*| \leq k \]

then $w(I*) \geq w(I)$ for all $I \in F$ with $|I| = |I*|$.

(i.e. a 'local' optimum is also a global optimum over independent sets of a fixed size.)

**Proof**

Suppose $I*$ satisfies 4.1 and $I* = \{e_1, \ldots, e_p\}$ where $w(e_i) \geq w(e_{i+1})$ for $1 \leq i < p$. Let $q = p \mod k$ and let $I_0 = \{e_1, \ldots, e_q\}$, $I_t = \{e_q+(t-1)k+1, \ldots, e_{q+tk}\}$ for $t = 1, \ldots, r = \lfloor p/k \rfloor$.

Let $J = \{f_1, \ldots, f_p\} \in F$ where $w(f_i) \geq w(f_{i+1})$ for $1 \leq i < p$ and let $J_0, J_1, \ldots, J_r$ be defined similarly. We show $w(I*) \geq w(J)$ by showing that $w(I_t) \geq w(J_t)$ for $t = 0, 1, \ldots, r$. Suppose to the contrary that $w(I_s) < w(J_s)$. Let

\[A = \bigcup_{t=0}^{s-1} I_t \quad (= \emptyset \text{ if } s = 0)\]

Let $B = \bigcup_{t=1}^{s} J_t$. Now either trivially if $s = 0$ or from axiom 2.2 if $s > 0$ there exists $Z \in B - A$ such that $|Z| = |I_s|$ and $C = A \cup Z \in F$. Note that $w(Z) > w(I_s)$. Let $D$ be obtained by successively using 2.2 to augment $C$, $k$ elements at a time from $I* - C$ until a set of size $p$ is reached. We need only note that $w(D) > w(I*)$ and yet $|D - I*| \leq k$.

We paraphrase 4.1 by saying that 'in an $M_k$-matroid k-optimality implies optimality. When $k = 1$ there is the converse result that 1-optimality only in matroids. For $k \geq 2$ this is not the case. We have given a great deal of (unsuccessful) thought to the problem of characterising when theorem 4.1 holds.
For $k=2$ we note that theorem 4.1 holds when $F = F_1 \cap F_2$ where $(E, F_1)$ is an arbitrary matroid and $F_2 = \{ I \subseteq E : |I \cap S| \leq p \}$ for some fixed $p$ and $S \subseteq I$ — a particularly simple matroid intersection. See Glover and Klingman [5].
REFERENCES


