

Adding random edges to create the square of a Hamilton cycle

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Abstract

We consider how many random edges need to be added to a graph of order n with minimum degree αn in order that it contains the square of a Hamilton cycle w.h.p..

1 Introduction

By the k th power of a Hamilton cycle, we mean a permutation (bijection) $\pi : [n] \rightarrow [n]$ such that $\{\pi(i), \pi(j)\} \in E(G)$ whenever $i < j \leq i + k$. (Here $i + k$ is to be taken as $i + k - n$ if $i + k \geq n + 1$.) Hamilton cycles have long been studied in the context of random graphs (see, e.g., [1, 3, 6, 11]). Powers of Hamilton cycles are less well-studied and much less is known about them.

Kühn and Osthus [8] observed that for $k \geq 3$, $p = \frac{1}{n^{1/k}}$ is the coarse threshold for the existence of the k th power of a Hamilton cycle in $G_{n,p}$. This comes directly from a result of Riordan [12]. For $k = 2$ they gave a bound of $p \geq n^{-1/2+\varepsilon}$ (for any $\varepsilon > 0$) being sufficient for the existence of the square of a Hamilton cycle w.h.p.. This result was improved by Nenadov and Škorić [10] to $p \geq \frac{C \log^4 n}{\sqrt{n}}$ (C is a positive constant) being sufficient for the existence of the square of a Hamilton cycle.

In this paper we consider a problem related to the Posá-Seymour conjecture, which states that every graph G on n vertices with minimum degree at least $kn/(k+1)$ contains the k th power of a Hamilton cycle. This conjecture was proved for large enough n by Komlós, Sarközy and Szemerédi [7]. Bohman, Frieze and Martin [2] considered the question of how many random edges need to be added to a graph with minimum degree αn with $0 < \alpha < 1/2$ in order that it is Hamiltonian w.h.p.. They showed that $(30 \log \alpha^{-1} + 13)n$ random edges are sufficient.

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The following theorem extends their result to the square of a Hamilton cycle. For graphs $G = (V, E)$ and $X = (V, F)$ we define a graph $G + X$ on vertex set V with edge set $E \cup F$.

Theorem 1 *For every constant $\alpha > 1/2$ there exists a sufficiently large $K = K(\alpha)$ such that the following holds. Let G be a graph of order n which has minimum degree at least αn . Let X denote a set of randomly chosen edges. Then w.h.p. $\Gamma = G + X$ contains the square of a Hamilton cycle, provided that*

$$|X| \geq Kn^{4/3} \log^{1/3} n.$$

Clearly $n^{4/3}$ is less than the $n^{3/2}$ needed if all edges are random.

2 Proof of Theorem 1

2.1 Preliminaries

It will be convenient for the computations to assume that the edges X will be given as $X = X_1 \cup X_2 \cup X_3$ where each of the sets in this partition are independent random subsets of $E(K_n)$ where each edge is independently included with probability $p = \frac{K \log^{1/3} n}{n^{2/3}}$ for some large positive constant K .

Assume first that $n = 2m$ is even. It follows from Erdős and Rényi [4] that w.h.p. the edges X_1 contain a perfect matching M . By symmetry, it will be a random matching of K_n that is independent of G . It can therefore be derived from a random permutation $\pi = (z_1, z_2, \dots, z_n)$ via $M = \{e_1, e_2, \dots, e_m\}$ where $e_i = \{z_{2i-1}, z_{2i}\}$, $i = 1, 2, \dots, m$.

Now for any graph H on vertex set $V(G)$, define a graph $\Pi(H)$ with vertex set M and an edge $\{e, f\}$, $e, f \in M$ whenever the subgraph $H_{e,f}$ of H induced by the four vertices in $e \cup f$ is K_4 . Let $\Gamma_1 = \Pi(G + X_1)$. We argue that

$$\text{w.h.p., } \Gamma_1 \text{ has minimum degree at least } \beta_1 n \text{ where } \beta_1 = (2\alpha - 1)^3/2. \quad (1)$$

To see this consider a fixed edge $e = \{x, y\} \in \Gamma_1$. Let $N(a)$ denote the set of neighbors of vertex a in G and note that $|N(x) \cap N(y)| \geq (2\alpha - 1)n$. The probability that another edge $\{u, v\} \in \Gamma_1$ satisfies $u, v \in N(x) \cap N(y)$ is at least $(1 - o(1))(2\alpha - 1)^2$. Thus the degree d_e of edge e has expectation at least $(1 - o(1))(2\alpha - 1)^3 n$ in Γ_1 . Swapping a pair in permutation π can only change d_e by at most one. Applying a version of the Azuma-Hoeffding inequality (see for example McDiarmid [9] or Frieze and Pittel [5]) we see that $\Pr(d_e \leq (2\alpha - 1)^3 n/2) \leq e^{-\Omega((2\alpha - 1)^6 n)}$. This verifies (1), after inflating the probability bound by m . Note that only the edges of Γ_1 are used here.

Now let $\Gamma_2 = \Pi(G + X_1 + X_2)$.

Lemma 2 Γ_2 is connected w.h.p..

Proof It follows from (1) that Γ_1 has at most $1/\beta_1$ components and these are all of size at least $\beta_1 n$. Thus,

$$\Pr(\Gamma_2 \text{ is not connected}) \leq \frac{1}{\beta_1} \max_{\beta_1 n \leq s \leq m/2} (1 - p^3)^{s(\alpha n - 2s)/2} = o(1).$$

Indeed, fix a component S of Γ_1 . It will have size $s \in [\beta_1 n/2, m/2]$. For each $e = \{u, v\} \in S$ there are at least $(\alpha n - 2s)$ vertices T outside $\bigcup_{e \in S} e$. For each vertex $x \in T$ we have a matching edge $f = \{x, y\} \in \bar{S}$ such that e and f are joined by an edge (say $\{u, x\}$) from G . The term p^3 accounts for the probability that X_2 will provide another three edges ($\{u, y\}$, $\{v, x\}$ and $\{v, y\}$) to create a K_4 . We divide by two in $s(\alpha n - 2s)/2$ to account for there being two choices for $x \in f$. \square

2.2 2-paths

A *2-path* is a sequence of vertices $(x_1, x_2, \dots, x_{2k})$ such that (i) $(x_1, x_2, \dots, x_{2k})$ is a path in $G + X$, (ii) $\{x_{2i-1}, x_{2i}\} \in M$, $i = 1, 2, \dots, k$, and (iii) $\{x_i, x_{i+2}\}$ are edges of $\Gamma = G + X$ for $i = 1, 2, \dots, 2k - 2$ (see Figure 1).

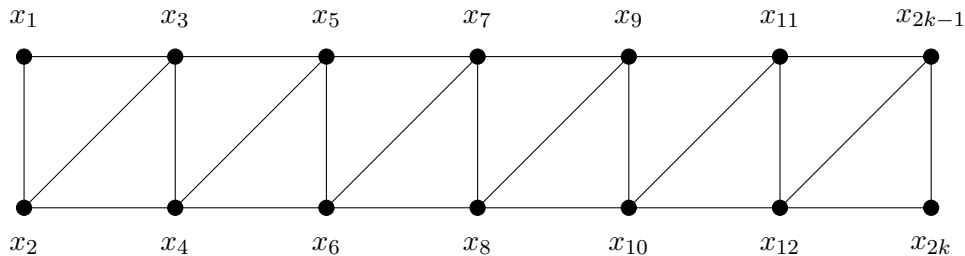


Figure 1: A 2-path for $k = 7$.

In a 2-path we refer to the edges $\{x_{2i-1}, x_{2i}\}$, $i = 1, 2, \dots, k$ as the *pillars*.

We now define a *rotation* with $\{x_1, x_2\}$ as the *fixed end* and $\{x_{2k-1}, x_{2k}\}$ as the *rotated end*. Suppose that for some $\ell \leq k - 2$ we have that $\{x_{2\ell-1}, x_{2k}\}$, $\{x_{2\ell}, x_{2k-1}\}$, $\{x_{2\ell}, x_{2k}\}$ are all edges of Γ . Then we obtain a new 2-path $(x_1, x_2, \dots, x_{2\ell}, x_{2k}, x_{2k-1}, \dots, x_{2\ell+1})$ (see Figure 2).

2.3 Algorithm ERA

Extension-Rotation algorithm

The algorithm begins by choosing an arbitrary edge $e \in M$ and letting path $P_1 = e$.

Basic Idea It proceeds in rounds. At the beginning of round k we will have a 2-path $P_k = (x_1, x_2, \dots, x_{2k})$. A round consists of the following: Let $Q_0 = P_k$ and then for $i = 1, 2, \dots$, if necessary, grow a set of paths Q_1, Q_2, \dots . Each Q_i is obtained from some Q_j , $j < i$ by a single rotation.

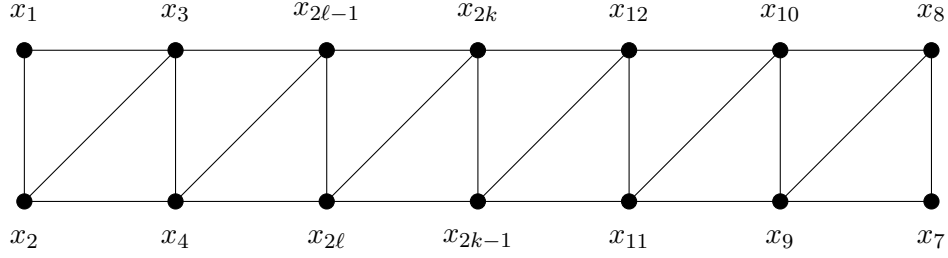


Figure 2: A rotated path for $k = 7$ and $\ell = 3$.

We continue until either we make a simple extension (defined below) or we make a cycle extension (defined below) or fail.

Simple Extensions The process is curtailed if at any point the procedure generates a path $P = (y_1, y_2, \dots, y_{2k})$ and an edge $\{u, v\} \in M$ disjoint from $V(P)$ such that $(y_1, y_2, \dots, y_{2k}, u, v)$ is a 2-path. In which case we can extend our current 2-path to one of length $2k + 2$ and end the round. We call this a *simple extension*.

Cycle extensions If we do not find a simple extension, then we see if there is a path $P = (y_1, y_2, \dots, y_{2k}) \in \mathcal{P}_L$ such that Γ contains the path $(y_{2k-1}, y_1, y_{2k}, y_2)$ (see Figure 3).

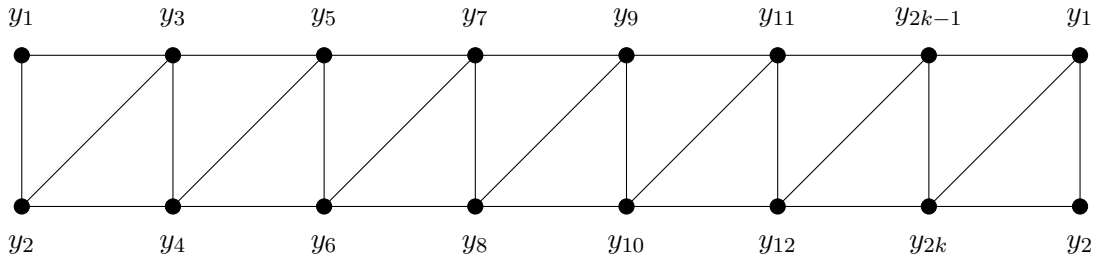


Figure 3: Closing a 2-path with $(y_{2k-1}, y_1, y_{2k}, y_2)$.

We say that we *close the path* to create a cycle $C = (y_1, y_2, \dots, y_{2k}, y_1)$. If $k = n/2$ then we have found the square of a Hamilton cycle. Otherwise, we seek a *cycle extension*. By this we mean that find an edge $\{u, v\} \in M$ disjoint from $V(P)$ such that and $1 \leq \ell < k$ such that $G + X$ contains the path $(y_{2\ell-1}, u, y_{2\ell}, v)$. In which case we now have the 2-path $(y_{2\ell+1}, y_{2\ell+2}, \dots, y_{2k}, y_{2k-1}, y_1, y_2, \dots, y_{2\ell-1}, y_{2\ell}, u, v)$. We call this a *cycle extension* (see Figure 4). If no such pair $\ell, \{u, v\}$ exists then we fail.

We can use all edges of Γ at any stage of the algorithm. However, in the description below, we only mention edges that are needed in the analysis. Here we rely on the fact that adding edges will not prevent a successful execution of ERA.

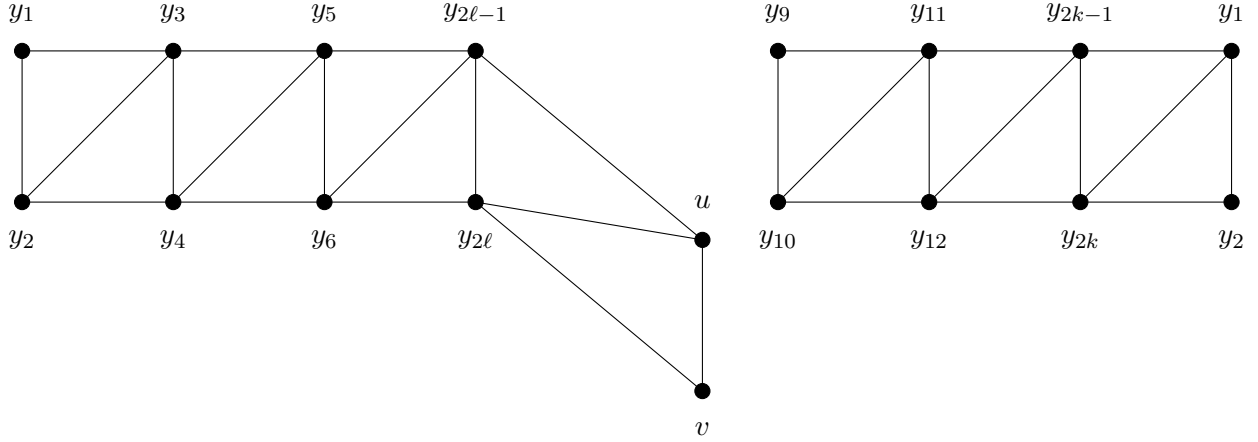


Figure 4: A cycle extension.

Step 1 We start a round with $P_k = (x_1, x_2, \dots, x_{2k})$. We then do a set of rotations with $e = \{x_1, x_2\}$ as the fixed end, one for each neighbor of $\{x_{2k-1}, x_{2k}\}$ in Γ_1 . Assuming there are no simple extensions we generate a set of 2-paths $Q_1, Q_2, \dots, Q_L, L \geq \beta_1 n$. The end pillar of P_i , other than $\{x_1, x_2\}$, will be denoted by e_i for $i = 1, 2, \dots, L$.

Step 2 After this, we take each Q_i in turn and do a set of rotations with e_i as the fixed end and e as the rotated end, using the edges of Γ_2 for this purpose.

Step 3 If we fail to obtain a simple extension, then we use the Γ edges to look for a cycle extension, using all of the 2-paths generated for this task.

2.4 Analysis of ERA

Lemma 3 *W.h.p. algorithm ERA succeeds in finding the square of a Hamilton cycle.*

Proof We argue that w.h.p. we can always find an X_3 edge to close a path if there is no simple extension. Let \mathcal{P}_k be the set of 2-paths generated in round k (Step 1 and 2). Thus,

$$|\mathcal{P}_k| \geq L \cdot L/2 \geq \beta_1^2 n^2 / 2$$

and consequently

$$\Pr(\text{No path of } \mathcal{P}_k \text{ can be closed}) \leq (1 - p^3)^{\beta_1^2 n^2 / 2} \leq e^{-\frac{1}{2} \beta_1^2 K^3 \log n}.$$

Since there at most $n/2$ rounds we see that w.h.p. there is at least one path in a round that can be closed, if needed.

Having closed a 2-path, the existence of $\ell, \{u, v\}$ follows from the connectivity of Γ_2 , see Lemma 2. \square

When $n = 2m + 1$ is odd, we use extra $Kn^{4/3} \log^{1/3} n$ edges X_4 (chosen independently from X_1 , X_2 and X_3) to find a subgraph as in Figure 5.

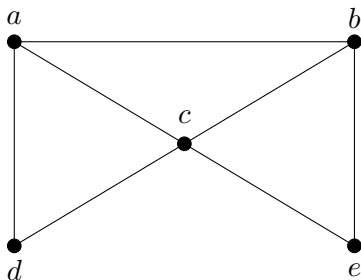


Figure 5: A graph used for n odd.

We can then use a, d and b, e as pillars and basically proceed as in the even case, making sure to avoid breaking up this subgraph and follow its vertices as (d, a, c, b, e) .

2.5 A lower bound

It is as well to consider lower bounds on the number of random edges needs to add to a graph G to obtain the k th power of Hamilton cycle. For this we consider the complete bipartite graph $G = K_{s,t}$ with bipartition A, B and where $s = |A| = \alpha n$ and $s + t = n$. We will be thinking here of the case where α is a small constant and so it does not fit exactly into the assumptions of Theorem 1. In $K_{s,t}$ the lower bound is much less than $n/2$ and have no lower bounds for the case where the minimum degree significantly exceeds $n/2$.

We can associate a sequence σ_H of length n over the alphabet $\{A, B\}$ with a Hamilton cycle H in $K_{s,t}$. The i th symbol will be an A if and only if the i th vertex of the cycle is in A . Only AB edges are in G and it is not difficult to show by examining σ_H that at most $2ks$ of the edges of H can be of this type. It follows that if we add edges to G with probability p then the expected number of k th powers will be at most $n!p^{k(n-2s)}$. Thus we require $p \geq n^{-1/k(1-2\alpha)}$ or at least $n^{2-1/k(1-2\alpha)}$ random edges. In particular, for $k = 2$ this implies that we need $n^{2-1/(2-4\alpha)}$ random edges, which for small α yields $n^{3/2-O(\alpha)}$. This is close to optimal, since $n^{3/2+o(1)}$ is the trivial upper bound.

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