Square of a Hamilton cycle in a random graph

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Abstract

We show that the threshold for the random graph $G_{n,p}$ to contain the square of a Hamilton cycle is $p = \frac{1}{\sqrt{n}}$. This improves the previous results of Kühn and Osthus and also Nenadov and Škorić. In addition we consider how many random edges need to be added to a graph of order n with minimum degree αn in order that it contains the square of a Hamilton cycle w.h.p.

1 Introduction

Let us recall a few classic models of random graphs. The *random graph* $G_{n,m}$ is the random graph G with vertex set $[n]$ and m random edges. The *random graph* $G_{n,p}$ is the random graph G with vertex set $[n]$ in which each of the $\binom{n}{2}$ edges of the complete graph K_n is included independently with probability p. Similarly, the *random digraph* $D_{n,m}$ is the random digraph D with vertex set $[n]$ and m random edges. The *random digraph* $D_{n,p}$ is the random digraph D with vertex set $[n]$ in which each of the $n(n-1)$ edges of the complete digraph $\vec{K}_n$ is included independently with probability p. We say that a sequence of events $\mathcal{E}_n$ in a probability space holds *with high probability* (or *w.h.p.*) if the probability that $\mathcal{E}_n$ holds tends to 1 as $n \to \infty$. Throughout this note all logarithms are natural (base e) and all asymptotics are taken in n.

By the *kth power of a Hamilton cycle*, we mean a permutation (bijection) $\pi : [n] \to [n]$ such that $\{\pi(i), \pi(j)\} \in E(G)$ whenever $i < j \leq i + k$. (Here $i + k$ is to be taken as $i + k - n$ if $i + k \geq n + 1$.) Hamilton cycles have long been studied in the context of random graphs (see, e.g., [1, 3, 7, 13]). Powers of Hamilton cycles are less well-studied and much less is known about

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them. Since the kth power of a Hamilton cycle contains kn edges, we can see that if X_k denotes the number of copies of such in $G_{n,p}$, then by using Stirling's formula,

$$\mathbf{E}(X_k) = n!p^{kn} \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n p^{kn} = \sqrt{2\pi n}\left(\frac{np^k}{e}\right)^n$$

and so $\mathbf{E}(X_k) \to 0$ if $p \le \left(\frac{(1-\varepsilon)e}{n}\right)^{1/k}$ for any constant $\varepsilon > 0$. Thus, if $p \le \left(\frac{(1-\varepsilon)e}{n}\right)^{1/k}$, then w.h.p. $G_{n,p}$ contains no kth power of a Hamilton cycle.

Kühn and Osthus [9] observed that for $k \ge 3$, $p = \frac{1}{n^{1/k}}$ is the coarse threshold for the existence of the kth power of a Hamilton cycle in $G_{n,p}$. This comes directly from a result of Riordan [14]. For $k = 2$ they gave a bound of $p \ge n^{-1/2+\varepsilon}$ (for any $\varepsilon > 0$) being sufficient for the existence of the square of a Hamilton cycle w.h.p.. This result was improved by Nenadov and Škorić [11] to $p \ge \frac{C\log^4 n}{\sqrt{n}}$ (C is a positive constant) being sufficient for the existence of the square of a Hamilton cycle. Here we improve this result by decreasing the lower bound to $\frac{\omega}{\sqrt{n}}$, where $\omega \to \infty$. As a matter of fact we prove a stronger result in the $D_{n,m}$ model.

Theorem 1 *If $\omega \to \infty$ and $m = \omega n^{3/2}$, then w.h.p. $D_{n,m}$ contains the square of a Hamilton cycle.*

This theorem also implies that the same holds in $D_{n,p}$, $G_{n,m}$ and $G_{n,p}$ with $p = \frac{\omega}{\sqrt{n}}$. We will give details of this implication at the end of Section 2. In the argument, we replace ω by monotone functions of ω, which is clearly acceptable.

The proof is based on the second moment method. Let us emphasize that applying the second moment method to $G_{n,p}$ or $D_{n,p}$ fails.

In this paper we also consider a problem related to the Posá-Seymour conjecture, which states that every graph G on n vertices with minimum degree at least $kn/(k+1)$ contains the kth power of a Hamilton cycle. This conjecture was proved for large enough n by Komlós, Sarközy and Szemerédi [8]. Bohman, Frieze and Martin [2] considered the question of how many random edges need to be added to a dense graph in order that it is Hamiltonian w.h.p. The following theorem extends their result to the square of a Hamilton cycle. For graphs $G = (V, E)$ and $X = (V, F)$ we define a graph $G + X$ on vertex set V with edge set $E \cup F$.

Theorem 2 *Let G be a graph of order n which has minimum degree at least αn for some absolute constant $\alpha > 1/2$. Let X denote a set of randomly chosen edges. Then w.h.p. $G + X$ contains the square of a Hamilton cycle, provided that*

$$|X| \ge Kn^{4/3}\log^{1/3} n.$$

Here we need $K = K(\alpha)$ to be sufficiently large.

Clearly $n^{4/3}$ is less than the $n^{3/2}$ needed if all edges are random.

2 Outline proof of Theorem 1

We will consider a slightly different model $D^*_{n,2m}$. We generate $D^*_{n,2m}$, where $m = \omega n^{3/2}$, by first choosing m random directed red edges and then choosing m random directed blue edges that are independent of the red edges (so an edge may be both red and blue). This is not a problem for the analysis and no edge will appear twice in the square of a Hamilton cycle. Also, after removing the colors, we see that we have a multigraph. A simple calculation shows that w.h.p. we have $O(\omega^2 n)$ repeated edges and we will assume that ω grows slowly. Thus we will have proved a property of $D_{n,2m-O(\omega^2 n)}$.

We say that a permutation π of $[n]$ in $D^*_{n,2m}$ is *square inducing* if $D^*_{n,2m}$ contains red edges $(\pi(i), \pi(i+1))$ and blue edges $(\pi(i), \pi(i+2))$ for each $i \in [n]$.

We let $\mathcal{H}_\pi$ be the event that permutation π is square inducing in $D^*_{n,2m}$. Let X be the random variable that counts the number of square inducing permutations.

$$\mathbf{E}(X) = \sum_\pi \mathbf{Pr}(\mathcal{H}_\pi) = n! \left(\frac{\binom{N-n}{m-n}}{\binom{N}{m}} \right)^2 = n! \prod_{i=0}^{n-1} \left(\frac{m-i}{N-i} \right)^2 = n! \left(\frac{m}{N} \right)^{2n} e^{-o(n^{1/2})},$$

which clearly goes to infinity.

We will prove that

$$\frac{\mathbf{E}(X^2)}{\mathbf{E}(X)^2} \leq f(\omega), \tag{1}$$

where $f(\omega) = e^{O(\omega^2)}$ for slowly growing ω.

The Paley-Zygmund inequality [12] implies that for $0 < \theta < 1$ we have

$$\mathbf{Pr}(X \geq \theta \mathbf{E}(X)) \geq (1-\theta)^2 \frac{\mathbf{E}(X)^2}{\mathbf{E}(X^2)}.$$

So, with $\theta = 1/2$ we can deduce that

$$\mathbf{Pr}(X > 0) \geq \frac{1}{4f(\omega)}.$$

Now let $p = \frac{2m}{N}$. Then we have, where $E_{n,p}$ is the set of edges of $D_{n,p}$,

$$\begin{aligned}
\mathbf{Pr}(X > 0 \text{ in } D_{n,p}) &\geq \mathbf{Pr}(X > 0 \text{ in } D_{n,p} \mid |E_{n,p}| \geq m) \mathbf{Pr}(|E_{n,p}| \geq m) \\
&\geq \sum_{m' \geq m} \mathbf{Pr}(X > 0 \text{ in } D_{n,m'}) - \mathbf{Pr}(|E_{n,p}| < m) \\
&\geq \frac{1}{4f(\omega)} - \mathbf{Pr}(|E_{n,p}| < m) \\
&\geq \frac{1}{4f(\omega)} - e^{-m/10} \\
&\geq \frac{1}{5f(\omega)}.
\end{aligned}$$

Now let $\omega_1 = \omega f(\omega)$ and let $m_1 = \omega_1 n^{3/2}$. Define p_1 by $p_1 = 1 - (1-p)^{\omega_1}$ so that D_{n,p_1} can be expressed as the union of ω_1 independent copies of $D_{n,p}$. Note that $p_1 \le \omega_1 p$. Then we have

$$\mathbf{Pr}(X > 0 \text{ in } D_{n,p_1}) \ge \left(1 - \frac{1}{5f(\omega)}\right)^{\omega_1} = 1 - o(1).$$

The monotonicity of containing the square of a Hamilton cycle implies that D_{n,m_1} contains the square of a Hamilton cycle w.h.p. Here $m_1 = Np_1 \le \omega f(\omega) n^{3/2}$ and this clearly implies Theorem 1, after replacing ω by $\omega f(\omega)$.

To prove the result in $G_{n,p}$ we let $p_2 = 1 - (1-p_1)^2$ and then we see that G_{n,p_1} is the graph underlying D_{n,p_2} and the square of a Hamilton cycle in D_{n,p_2} induces one in G_{n,p_1}. Finally, we obtain the result for $G_{n,m}$ by monotonicity.

3 Proof of (1)

Fix a square inducing permutation π, and let $H(\pi)$ be the square hamilton cycle induced by π. $H(\pi)$ consists of the red "distance 1 edges" $(\pi(i), \pi(i+1))$ and blue "distance 2 edges" $(\pi(i), \pi(i+2))$. We consider some other square inducing permutation $\widehat{\pi}$, and say that π shares a distance $k \in \{1,2\}$ edge with $\widehat{\pi}$ if $(\pi(i), \pi(i+k)) = (\widehat{\pi}(j), \widehat{\pi}(j+k))$ for some i, j. For F a set of distance 2 edges in $H(\pi) \cap H(\widehat{\pi})$ and F' in $H(\widehat{\pi})$ a set of paths ("paths" are all directed paths in this paper) of distance 1 edges in $\widehat{\pi}$, we say that F' *lies underneath* F if F is precisely the set of edges $(\widehat{\pi}(i), \widehat{\pi}(i+2))$ for indices i such that $(\widehat{\pi}(i), \widehat{\pi}(i+1))$ and $(\widehat{\pi}(i+1), \widehat{\pi}(i+2)$ are consecutive edges of a path in F'. A path in F' is a maximal sequence of the form $(\widehat{\pi}(l), \widehat{\pi}(l+1), \widehat{\pi}(l+2), \widehat{\pi}(l+3), \ldots) = (\pi(i), \pi(j), \pi(i+2), \pi(j+2), \ldots)$ that ends in $\pi(i+2s)$ or $\pi(j+2s)$ of vertices that are consecutive in $H(\widehat{\pi})$. These paths in F' may share endpoints. For example, a sequence $\pi(i), \pi(j), \pi(i+2), \pi(k), \pi(i+4)$ is considered to be two paths of length two if $k \neq j+2$. Note that every set F has a unique corresponding set of paths F' lying underneath F. Let $N^*(a,b,c,d)$ be the number of square inducing permutations $\widehat{\pi}$ such that:

- **(a)** $H(\pi), H(\widehat{\pi})$ share b distance one edges (i.e. red edges), arranged in a common paths. These paths are obviously red.
- **(b)** Let F be the set of shared (blue) distance 2 edges in $H(\pi) \cap H(\widehat{\pi})$, not including the shared (blue) distance 2 edges that arise from 2-paths of shared (red) distance 1 edges. That means not including edges of the form $(\pi(i), \pi(i+2))$ such that both $(\pi(i), \pi(i+1))$ and $(\pi(i+1), \pi(i+2))$ are shared edges. Then $|F| = d$ and the underlying set of paths F' has c paths and consequently $c+d$ edges.

We will first show that $N^*(a,b,c,d) \le N(a,b,c,d)$ for some explicit function $N(a,b,c,d)$. Note that we always have $0 \le a \le b \le n$ and $0 \le c \le d \le n$, which we assume below:

$$\frac{\mathbf{E}(X^2)}{\mathbf{E}(X)^2} \le \sum_b \sum_d \sum_a \sum_c \frac{n! N(a,b,c,d) \binom{N-2n+b}{m-2n+b}\binom{N-2n+d}{m-2n+d} / \binom{N}{m}^2}{\mathbf{E}(X)^2}$$

$$= \sum_b \sum_d \sum_a \sum_c \frac{N(a,b,c,d) \binom{N-2n+b}{m-n+b}\binom{N-2n+d}{m-2n+d}\binom{N}{m}^2}{n! \binom{N-n}{m-n}^4}. \tag{2}$$

Now observe that

$$\frac{\binom{N}{m}}{\binom{N-n}{m-n}} = \prod_{i=0}^{n-1} \frac{N-i}{m-i} = \left(\frac{N}{m}\right)^n \prod_{i=0}^{n-1} \left(1-\frac{i}{N}\right)\left(1-\frac{i}{m}\right)^{-1} \le \left(\frac{N}{m}\right)^n \exp\left\{\frac{n^2}{2m}\right\}$$

and

$$\frac{\binom{N-2n+b}{m-2n+b}}{\binom{N-n}{m-n}} = \prod_{i=0}^{n-b-1} \frac{m-(n+i)}{N-(n+i)} = \left(\frac{m}{N}\right)^{n-b} \prod_{i=0}^{n-b-1} \left(1-\frac{n+i}{m}\right)\left(1-\frac{n+i}{N}\right)^{-1}$$

$$\le_{\mathcal{O}} \left(\frac{m}{N}\right)^{n-b} \exp\left\{-\frac{3n^2-4nb}{2m}\right\},$$

where $f \le_{\mathcal{O}} g$ means $f = O(g)$. Consequently,

$$\frac{\binom{N-2n+b}{m-n+b}\binom{N-2n+d}{m-2n+d}\binom{N}{m}^2}{\binom{N-n}{m-n}^4} \le_{\mathcal{O}} \exp\left\{-\frac{2N}{m}\right\}\left(\frac{Ne^{2n/m}}{m}\right)^{b+d}.$$

Now we define an upper bound $N(a,b,c,d)$ for any $0 \le a \le b \le n$ and $0 \le c \le d \le n$. If $a = c = 0$ (implying $b = d = 0$), then we take $N(0,0,0,0) = n!$. Otherwise,

$$N(a,b,c,d) = \frac{n}{a+c}\binom{b-1}{a-1}\binom{d-1}{c-1}\binom{a+c}{a} \times \sum_{g_0=0}^{a+c} \binom{a+c}{g_0}\binom{n-b-c-d-1}{a+c-g_0-1} n^{a+2c-g_0}(n-a-b-2c-d+g_0)!,$$

where we use the convention $\binom{x}{-1} = 1$ for any $x > 0$. Here g_0 is the number of zero length gaps, see **S3** below.

We will now explain why $N^*(a,b,c,d) \le N(a,b,c,d)$. First recall that the number of integer solutions of

$$x_1 + \cdots + x_\ell = s \tag{3}$$

with $x_i \ge k$ (for each $1 \le i \le \ell$) is exactly $\binom{s-\ell(k-1)-1}{\ell-1}$. Obviously $N^*(0,0,0,0) \le n! = N(0,0,0,0)$. The remaining case we prove through the following steps:

S1 We choose a many positive lengths summing to b obtaining the lengths for paths in (a). This can be done in $\binom{b-1}{a-1}$ ways. (Apply (3) with $\ell = a$, $s = b$ and $k = 1$.)

S2 Similarly we choose c many lengths that are at least 2, summing to $c+d$ obtaining the lengths for underlying set of paths in (b). We have exactly $\binom{d-1}{c-1}$ choices. (Apply (3) with $\ell = c$, $s = c+d$ and $k = 2$.)

S3 These paths will have gaps (possibly zero length gaps) between. Let g_0 be the number of gaps of length zero. First we choose which gaps have zero length: $\binom{a+c}{g_0}$ ways. Note that it is actually not possible to have a zero gap between two paths (a), so we overcount here. It is possible to have a zero gap between two blue paths, or between a red and a blue path.

S4 Now we choose $a+c-g_0$ many positive lengths for gaps, summing to $n-b-c-d$. This gives $\binom{n-b-c-d-1}{a+c-g_0-1}$ many ways. (Apply (3) with $\ell = a+c-g_0$, $s = n-b-c-d$ and $k = 1$.)

S5 Now we specify the order of our red and blue paths in $\widehat{\pi}$. We already have an ordered set of a many lengths of red paths, and another ordered set of c lengths of blue paths. Thus the number of ways to specify an ordering of these $a+c$ many paths together is $\binom{a+c}{a}$.

S6 We choose a "starting index" (n choices) for the first of our $a+c$ many paths. Once this choice is made we have specified the positions of all the paths and gaps in $\widehat{\pi}$. But each possible configuration is counted $a+c$ many times. So for this part we multiply by $\frac{n}{a+c}$.

Let us describe what we know about $\widehat{\pi}$ at this point. By "starting index" we mean we choose an index i such that the first of our $a+c$ paths, if it has say length L, will be the path $\widehat{\pi}(i), \widehat{\pi}(i+1), \ldots, \widehat{\pi}(i+L)$. Then the first gap, say if it has length L' will be $\widehat{\pi}(i+L), \ldots, \widehat{\pi}(i+L+L')$. Note that we have not yet specified any of the actual values of $\widehat{\pi}(j)$ for any j. We will start doing that next.

S7 We choose the value that $\widehat{\pi}$ takes at the first index of each path in (a) and each underlying path in (b), except for those paths which come immediately after a gap of length zero. Since there are g_0 zero length gaps we have n^{a+c-g_0} choices. Now we choose the second vertex for each underlying paths having n^c choices.

For example if we already know a certain path (which does not come immediately after a zero length gap) in (a) is $\widehat{\pi}(j), \ldots, \widehat{\pi}(j+L)$ then we choose the value of $\widehat{\pi}(j)$. Note that there is some index j' such that $\widehat{\pi}(j) = \pi(j')$, and since this is a path of shared edges we must have $\widehat{\pi}(j+t) = \pi(j'+t)$ for $0 \le t \le L$.

Similarly if we already know a certain path (which does not come after a zero length gap) in (b) is $\widehat{\pi}(j), \ldots, \widehat{\pi}(j+L)$ then we choose the value of $\widehat{\pi}(j), \widehat{\pi}(j+1)$. Note that there are some indices j', j'' such that $\widehat{\pi}(j) = \pi(j'), \widehat{\pi}(j+1) = \pi(j'')$, and so $\widehat{\pi}(j), \widehat{\pi}(j+1), \widehat{\pi}(j+2), \widehat{\pi}(j+3) \ldots = \pi(j'), \pi(j''), \pi(j'+2), \pi(j''+2) \ldots$.

For paths of type (a) or (b) that do come immediately after a zero gap, the value of $\widehat{\pi}$ at the first index is already determined by the path that came before. However we always choose the value of $\widehat{\pi}$ at the second index of each path in (b).

S8 Finally we choose the values of $\widehat{\pi}$ at all of the indices that are not in any of the paths from (a) and (b). There are $a+b+2c+d-g_0$ many vertices on these paths.

Observe that in (2) the first term ($b=d=a=c=0$) is negligible. Thus, from now on we assume that $a+c\ge 1$ and get

$$\begin{aligned}\frac{\mathbf{E}(X^2)}{\mathbf{E}(X)^2} &\le \sum_b\sum_d\sum_a\sum_c \frac{N(a,b,c,d)\binom{N-2n+b}{m-n+b}\binom{N-2n+d}{m-2n+d}\binom{N}{m}^2}{n!\binom{N-n}{m-n}^4}\\ &\le_{\mathcal{O}} \frac{1}{n!}\exp\left\{-\frac{2N}{m}\right\}\sum_b\sum_d\sum_a\sum_c\sum_{g_0}\frac{n}{a+c}\binom{b-1}{a-1}\binom{d-1}{c-1}\binom{a+c}{a}\binom{a+c}{g_0}\\ &\qquad\times\binom{n-b-c-d-1}{a+c-g_0-1}n^{a+2c-g_0}(n-a-b-2c-d+g_0)!\left(\frac{Ne^{2n/m}}{m}\right)^{b+d}. \qquad (4)\end{aligned}$$

Now let $\delta=\omega/n^{1/2}$ and consider two cases.

Case 1: $b\le\delta n$ and $d\le\delta n$

Since $a\le b$ and $c\le d$, we have $a,c\le\delta n$ and $g_0\le a+c\le 2\delta n$. Therefore,

$$\begin{aligned}\frac{n^{a+2c-g_0}(n-a-b-2c-d+g_0)!}{(n-b-d)!} &= \frac{n^{a+2c-g_0}}{(n-b-d)_{a+2c-g_0}} \le \frac{n^{a+2c-g_0}}{(n-5\delta n)^{a+2c-g_0}}\\ &= \left(\frac{1}{1-5\delta}\right)^{a+2c-g_0} \le \left(\frac{1}{1-5\delta}\right)^{3\delta n} \le e^{20\delta^2 n}=e^{20\omega^2}\end{aligned}$$

and so (4) is at most

$$\begin{aligned}&e^{20\omega^2}\frac{1}{n!}\exp\left\{-\frac{2N}{m}\right\}\sum_b\sum_d\sum_a\sum_c\sum_{g_0}\frac{n}{a+c}\binom{b-1}{a-1}\binom{d-1}{c-1}\binom{a+c}{a}\binom{a+c}{g_0}\\ &\qquad\times\binom{n-b-c-d-1}{a+c-g_0-1}(n-b-d)!\left(\frac{Ne^{2n/m}}{m}\right)^{b+d}.\end{aligned}$$

Now we use the Vandermonde identity to obtain

$$\sum_{g_0=0}^{a+c}\binom{a+c}{g_0}\binom{n-b-c-d-1}{a+c-g_0-1}=\binom{n+a-b-d-1}{a+c-1}$$

to get the bound

$$\begin{aligned}&e^{20\omega^2}\frac{1}{n!}\exp\left\{-\frac{2N}{m}\right\}\sum_b\sum_d\sum_a\sum_c\frac{n}{a+c}\binom{b-1}{a-1}\binom{d-1}{c-1}\binom{a+c}{a}\\ &\qquad\times\binom{n+a-b-d-1}{a+c-1}(n-b-d)!\left(\frac{Ne^{2n/m}}{m}\right)^{b+d}\\ &\qquad\qquad\le_{\mathcal{O}} e^{20\omega^2}\frac{1}{n!}\exp\left\{-\frac{2N}{m}\right\}\sum_b\sum_d\sum_a\sum_c\binom{b-1}{a-1}\binom{d-1}{c-1}\binom{a+c}{a}\\ &\qquad\qquad\qquad\times\binom{n+a-b-d-1}{a+c}(n-b-d)!\left(\frac{Ne^{2n/m}}{m}\right)^{b+d}. \qquad (5)\end{aligned}$$

Now note that

$$\sum_{c=0}^{d}\binom{d-1}{c-1}\binom{a+c}{a}\binom{n+a-b-d-1}{a+c} = \binom{n+a-b-d-1}{a}\sum_{c}\binom{d-1}{d-c}\binom{n-b-d-1}{c}$$
$$= \binom{n+a-b-d-1}{a}\binom{n-b-2}{d}.$$

Hence, (5) can be bounded by

$$e^{20\omega^2}\frac{1}{n!}\exp\left\{-\frac{2N}{m}\right\}\sum_b\sum_d\sum_a\binom{b-1}{a-1}\binom{n+a-b-d-1}{a}\binom{n-b}{d}(n-b-d)!\left(\frac{Ne^{2n/m}}{m}\right)^{b+d}$$
$$\leq e^{20\omega^2}\frac{1}{n!}\exp\left\{-\frac{2N}{m}\right\}\sum_b\sum_d\sum_a\binom{b-1}{a-1}\binom{n-d-1}{a}\binom{n-b}{d}(n-b-d)!\left(\frac{Ne^{2n/m}}{m}\right)^{b+d}$$
$$= e^{20\omega^2}\frac{1}{n!}\exp\left\{-\frac{2N}{m}\right\}\sum_b\sum_d\binom{n+b-d-2}{b}\binom{n-b}{d}(n-b-d)!\left(\frac{Ne^{2n/m}}{m}\right)^{b+d} \tag{6}$$

and note that since

$$\frac{(n+b-d-2)_b(n-b)_d(n-b-d)!}{n!} = \frac{(n+b-d-2)_b}{(n)_b} \leq \left(\frac{n+b}{n-b}\right)^b \leq \left(\frac{n+\delta n}{n-\delta n}\right)^{\delta n}$$
$$= \left(\frac{1+\delta}{1-\delta}\right)^{\delta n} \leq (1+3\delta)^{\delta n} \leq e^{3\omega^2}$$

we can say that (6) is at most

$$e^{23\omega^2}\exp\left\{-\frac{2N}{m}\right\}\sum_b\sum_d\frac{\left(\frac{Ne^{2n/m}}{m}\right)^{b+d}}{b!d!} \leq e^{23\omega^2}\exp\left\{-\frac{2N}{m}\right\}\sum_{b=0}^{\infty}\frac{\left(\frac{Ne^{2n/m}}{m}\right)^{b}}{b!}\sum_{d=0}^{\infty}\frac{\left(\frac{Ne^{2n/m}}{m}\right)^{d}}{d!}$$
$$= e^{23\omega^2}\exp\left\{-\frac{2N}{m}\right\}\exp\left\{\frac{Ne^{2n/m}}{m}\right\}\exp\left\{\frac{Ne^{2n/m}}{m}\right\}$$
$$= e^{23\omega^2}\exp\left\{\frac{2N}{m}\left(e^{2n/m}-1\right)\right\} = e^{23\omega^2}\exp\left\{O\left(\frac{Nn}{m^2}\right)\right\} \leq e^{25\omega^2}.$$

Case 2: $\max\{b,d\} \geq \delta n$

We will show that the term of (4)

$$\frac{1}{n!}\exp\left\{-\frac{2N}{m}\right\}\sum_a\sum_c\sum_{g_0}\frac{n}{a+c}\binom{b-1}{a-1}\binom{d-1}{c-1}\binom{a+c}{a}\binom{a+c}{g_0}$$
$$\times\binom{n-b-c-d-1}{a+c-g_0-1}n^{a+2c-g_0}(n-a-b-2c-d+g_0)!\left(\frac{Ne^{2n/m}}{m}\right)^{b+d} \tag{7}$$

is very small. First note that

$$\frac{n^{a+2c-g_0}(n-a-b-2c-d+g_0)!}{n!} \leq_{\mathcal{O}} \frac{n^{a+2c-g_0}\sqrt{n-a-b-2c-d+g_0}\left(\frac{n-a-b-2c-d+g_0}{e}\right)^{n-a-b-2c-d+g_0}}{\sqrt{n}\left(\frac{n}{e}\right)^n} \leq e^{O(b+d)}n^{-b-d}.$$

Thus, we have that (7) is at most

$$\begin{aligned}
&e^{O(b+d)}\left(\frac{N}{mn}\right)^{b+d}\sum_a\sum_c\sum_{g_0}\frac{n}{a+c}\binom{b-1}{a-1}\binom{d-1}{c-1}\binom{a+c}{a}\binom{a+c}{g_0}\binom{n-b-c-d-1}{a+c-g_0-1}\\
&= e^{O(b+d)}\left(\frac{N}{mn}\right)^{b+d}\sum_a\sum_c\frac{n}{a+c}\binom{b-1}{a-1}\binom{d-1}{c-1}\binom{a+c}{a}\binom{n+a-b-d-1}{a+c-1}\\
&= e^{O(b+d)}\left(\frac{N}{mn}\right)^{b+d}\sum_a\sum_c\binom{b-1}{a-1}\binom{d-1}{c-1}\binom{a+c}{a}\binom{n+a-b-d}{a+c}\\
&\leq e^{O(b+d)}\left(\frac{N}{mn}\right)^{b+d}\sum_a\sum_c\binom{b-1}{a-1}\binom{d-1}{c-1}\binom{a+c}{a}\binom{n-d}{a+c}\\
&= e^{O(b+d)}\left(\frac{N}{mn}\right)^{b+d}\sum_a\sum_c\binom{b-1}{a-1}\binom{d-1}{c-1}\binom{n-d-a}{c}\binom{n-d}{a}\\
&= e^{O(b+d)}\left(\frac{N}{mn}\right)^{b+d}\sum_a\binom{b-1}{a-1}\binom{n-a-1}{d}\binom{n-d}{a}\\
&\leq e^{O(b+d)}\left(\frac{N}{mn}\right)^{b+d}\sum_a\binom{b-1}{a-1}\binom{n}{d}\binom{n-d}{a}\\
&= e^{O(b+d)}\left(\frac{N}{mn}\right)^{b+d}\binom{n}{d}\binom{n+b-d-1}{b}\\
&\leq e^{O(b+d)}\left(\frac{N}{m}\right)^{b+d}\frac{1}{b!d!}\\
&\leq \left(\frac{NO(1)}{mb}\right)^b\left(\frac{NO(1)}{md}\right)^d
\end{aligned}$$

which is exponentially small. To see this, first note that the final expression is symmetric. Hence, without loss of generality we can assume that $b \geq \delta n$. Then

$$\left(\frac{NO(1)}{mb}\right)^b = \left(\frac{O(1)}{\omega^2}\right)^b = e^{-\Omega(b\log\omega)}$$

and since the function $f(x) = (C/x)^x$ is maximized at $x = C/e$ where we have $f(x) = e^{C/e}$, we may bound

$$\left(\frac{NO(1)}{md}\right)^d = e^{O(N/m)} = e^{O(b)}.$$

This completes the proof of (1) and hence Theorem 1.

4 Proof of Theorem 2

4.1 Preliminaries

It will be convenient for the computations to assume that the edges X will be given as $X = X_1 \cup X_2$ where each of the sets in this partition are independent random subsets of $E(K_n)$ where each edge is independently included with probability $p = \frac{K \log^{1/3} n}{n^{2/3}}$.

Assume first that $n = 2m$ is even. It follows from Erdős and Rényi [5] that w.h.p. the edges X_1 contain a perfect matching M. By symmetry, it will be a random matching of K_n that is independent of G. It can therefore be derived from a random permutation $\pi = (z_1, z_2, \ldots, z_n)$ via $M = \{e_1, e_2, \ldots, e_m\}$ where $e_i = \{z_{2i-1}, z_{2i}\}, i = 1, 2, \ldots, m$.

Now define a graph Γ with vertex set M and an edge $\{e, f\}, e, f \in M$ whenever the subgraph $H_{e,f}$ of $G + X_1$ induced by the four vertices in $e \cup f$ is K_4. We argue that

$$\text{W.h.p., } \Gamma \text{ has minimum degree at least } \beta_1 n \text{ where } \beta_1 = (2\alpha - 1)^2/3. \tag{8}$$

To see this consider a fixed edge $e = \{x, y\} \in M$. Let $N(a)$ denote the set of neighbors of vertex a in G and note that $|N(x) \cap N(y)| \geq (2\alpha - 1)n$. The probability that another edge $\{u, v\} \in M$ satisfies $u, v \in N(x) \cap N(y)$ is at least $(1 - o(1))(2\alpha - 1)^2/2$. Thus the degree d_e of edge e has expectation at least $(1 - o(1))(2\alpha - 1)n/2$ in Γ. Swapping a pair in permutation π can only change d_e by at most two. Applying a version of the Azuma-Hoeffding inequality (see for example McDiarmid [10] or Frieze and Pittel [6]) we see that $\mathbf{Pr}(d_e \leq (2\alpha - 1)^2 n/3) \leq e^{-\Omega((2\alpha - 1)^4 n)}$. This verifies (8), after inflating the probability bound by m. Note that only the $G + X_1$-edges of Γ are used here.

We prove next the following lemma.

Lemma 3 *Γ is connected w.h.p.*

Proof It follows from (8) that we only need to show there are no components S of size $s \in [\beta_1 n/2, m/2]$. But then

$$\begin{aligned}
\mathbf{Pr}(\exists \text{ component of size } s \in [\beta_1 n/2, m/2]) &\leq \sum_{s=\beta_1 n/2}^{m/2} \binom{n}{s} (1 - p^3)^{s(\alpha n - 2s)/2} \\
&\leq \sum_{s=\beta_1 n/2}^{n/4} \left(\frac{ne}{s}\right)^s e^{-s(\alpha n - 2s)p^3/2} \\
&\leq \sum_{s=\beta_1 n/2}^{n/4} \left(\frac{ne}{s} \cdot e^{-\frac{\beta_1}{4} K^3 (\alpha - \frac{1}{2}) \log n}\right)^s \\
&= o(1).
\end{aligned} \tag{9}$$

Explanation of (9)**:** For each $e \in S$ there are at least $(\alpha n - 2s)$ vertices T outside $\bigcup_{e \in S} e$. For each vertex $w \in T$ we have a matching edge $f \in \bar{S}$ such that e and f are joined by an edge

from G. The term p^3 accounts for the probability that X_2 will provide another three edges to create a K_4. We divide by two in $s(\alpha n - 2s)/2$ to account for there being two choices for $w \in f$.
□

4.2 2-paths

A *2-path* is a sequence of vertices $(x_1, x_2, \ldots, x_{2k})$ such that (i) $(x_1, x_2, \ldots, x_{2k})$ is a path in $G + X$, (ii) $\{x_{2i-1}, x_{2i}\} \in M$, $i = 1, 2, \ldots, k$, and (iii) $\{x_i, x_{i+2}\}$ are edges of $G + X$ for $i = 1, 2, \ldots, 2k-2$ (see Figure 1).

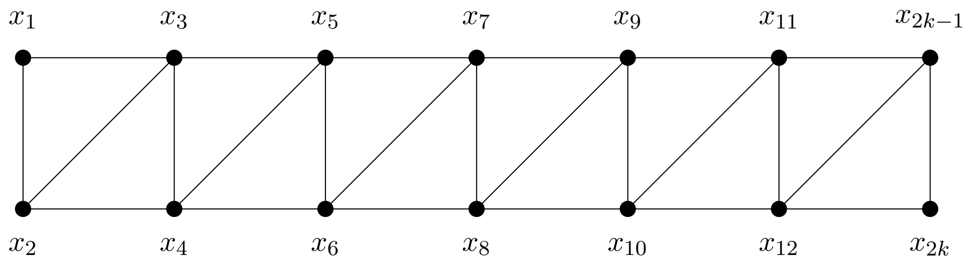

Figure 1: A 2-path for $k = 7$.

In a 2-path we refer to the edges $\{x_{2i-1}, x_{2i}\}$, $i = 1, 2, \ldots, k$ as the *pillars*.

We now define a *rotation* with $\{x_1, x_2\}$ as the *fixed end* and $\{x_{2k-1}, x_{2k}\}$ as the *rotated end*. Suppose that for some $\ell \leq k - 2$ we have that $\{x_{2\ell-1}, x_{2k}\}, \{x_{2\ell}, x_{2k-1}\}, \{x_{2\ell}, x_{2k}\}$ are all edges of $G + X$. Then we obtain a new 2-path $(x_1, x_2, \ldots, x_{2\ell}, x_{2k}, x_{2k-1}, \ldots, x_{2\ell+1})$ (see Figure 2).

4.3 Algorithm ERA

Extension-Rotation algorithm
The algorithm begins by choosing an arbirtrary edge $e \in M$ and letting path $P_1 = e$.

Basic Idea It proceeds in rounds. At the beginning of round k we will have a 2-path $P_k =$

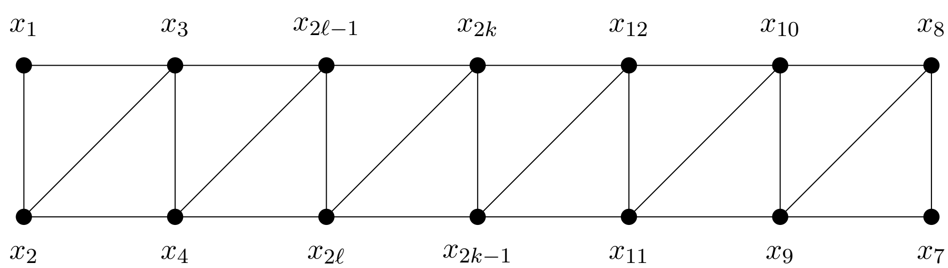

Figure 2: A rotated path for $k = 7$ and $\ell = 3$.

$(x_1, x_2, \ldots, x_{2k})$. A round consists of the following: Let $Q_0 = P_k$ and then for $i = 1, 2, \ldots$, if necessary, grow a set of paths $Q_1, Q_2, \ldots$. Each Q_i is obtained from some $Q_j, j < i$ by a single rotation.

We continue until either we make a simple extension (defined below) or we make a cycle extension (defined below) or fail.

Simple Extensions The process is curtailed if at any point the procedure generates a path $P = (y_1, y_2, \ldots, y_{2k})$ and an edge $\{u, v\} \in M$ disjoint from $V(P)$ such that $(y_1, y_2, \ldots, y_{2k}, u, v)$ is a 2-path. In which case we can extend our current 2-path to one of length $2k + 2$ and end the round. We call this a *simple extension.*

Cycle extensions If we do not find a simple extension, then we see if there is a path $P = (y_1, y_2, \ldots, y_{2k}) \in \mathcal{P}_L$ such that $G + X$ contains the path $(y_{2k-1}, y_1, y_{2k}, y_2)$ (see Figure 3).

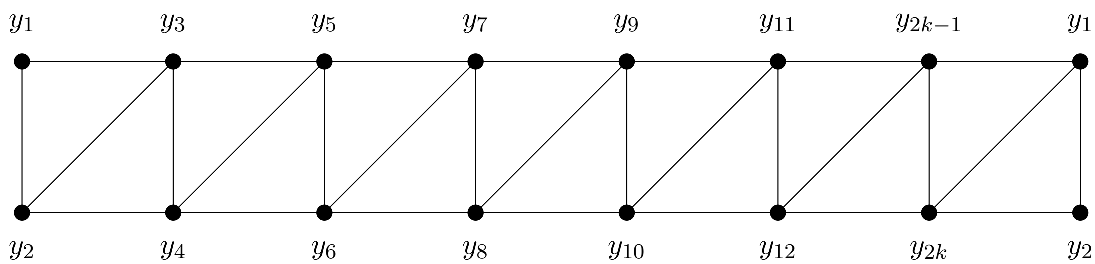

Figure 3: Closing a 2-path with $(y_{2k-1}, y_1, y_{2k}, y_2)$.

We say that we *close the path* to create a cycle $C = (y_1, y_2, \ldots, y_{2k}, y_1)$. If we find such an edge and $k = n/2$ then we have found the square of a Hamilton cycle. Otherwise, we seek a *cycle extension.* By this we mean that find an edge $\{u, v\} \in M$ disjoint from $V(P)$ such that and $1 \leq \ell < k$ such that $G + X$ contains the path $(y_{2\ell-1}, u, y_{2\ell}, v)$. In which case we now have the 2-path $(y_{2\ell+1}, y_{2\ell+2}, \ldots, y_{2k}, y_{2k-1}, y_1, y_2, \ldots, y_{2\ell-1}, y_{2\ell}, u, v)$. We call this a *cycle extension* (see Figure 4). If no such pair $\ell, \{u, v\}$ exists then we fail.

We need to be a little more precise about the order of rotations.

Step 1 We start a round with $P_k = (x_1, x_2, \ldots, x_{2k})$. We then do a set of rotations with $e = \{x_1, x_2\}$ as the fixed end, one for each neighbor of $\{x_{2k-1}, x_{2k}\}$ in Γ. We will use the edges of $G + X_1$ when we check for simple extensions. Assuming there are no simple extensions we generate a set of 2-paths $Q_1, Q_2, \ldots, Q_L, L \geq \beta_1 n$. The end pillar of P_i, other than $\{x_1, x_2\}$, will be denoted by e_i for $i = 1, 2, \ldots, L$.

Step 2 After this, we take each Q_i in turn and do a set of rotations with e_i as the fixed end and e as the rotated end, using the edges of $G + X_1$ for this purpose.

Step 3 If we fail to obtain a simple extension, then we use the $G + X$ edges to look for a cycle extension, using all of the 2-paths generated for this task.

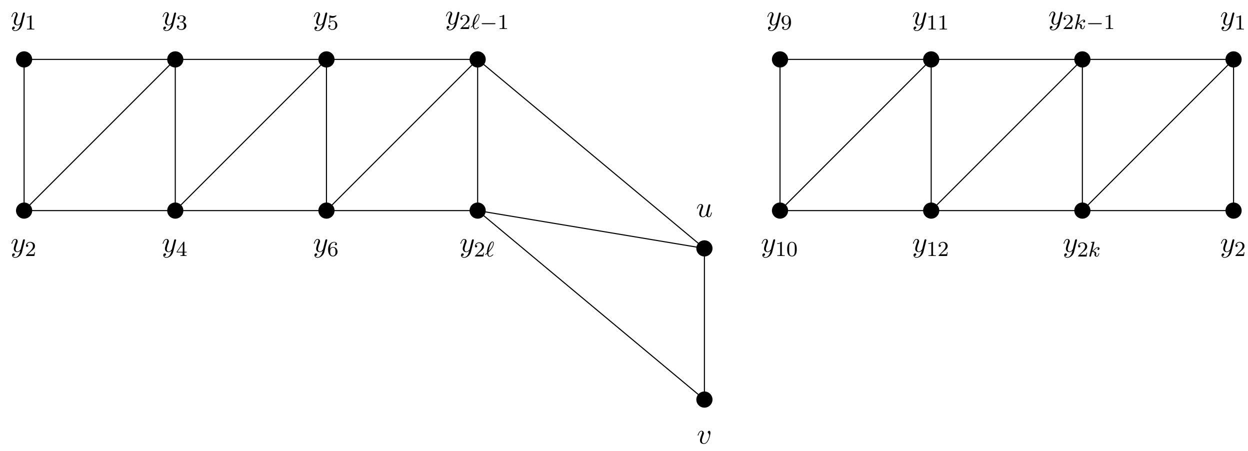

Figure 4: A cycle extension.

4.4 Analysis of ERA

Lemma 4 *W.h.p. algorithm* ERA *succeeds in finding the square of a Hamilton cycle.*

Proof We argue that w.h.p. we can always find an X_2 edge to close a path if there is no simple extension. For a fixed round we have from (8) that

$$\mathbf{Pr}(\text{No path of } \mathcal{P}_k \text{ can be closed}) \leq (1-p^3)^{\beta_1^2 n^2/2} \leq e^{-\frac{1}{2}\beta_1^2 K^3 \log n}. \tag{10}$$

Since there at most $n/2$ rounds we see that w.h.p. there is at least one path in a round that can be closed, if needed.

Having closed a 2-path, the existence of $\ell, \{u, v\}$ follows from the connectivity of Γ, see Lemma 3. $\square$

When $n = 2m+1$ is odd, we use extra $Kn^{4/3}\log^{1/3} n$ edges X_3 (chosen independently from X_1 and X_2) to find a subgraph as in Figure 5.

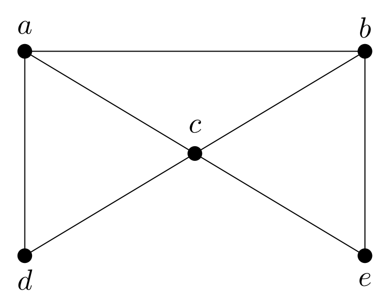

Figure 5: A graph used for n odd.

We can then use a, d and b, e as pillars and basically proceed as in the even case, making sure to avoid breaking up this subgraph and follow its vertices as (d, a, c, b, e).

4.5 A lower bound

It is as well to consider lower bounds on the number of random edges needs to add to a graph G to obtain the kth power of Hamilton cycle. For this we consider the complete bipartite graph $G = K_{s,t}$ with bipartition A, B and where $s = |A| = \alpha n$ and $s + t = n$. We will be thinking here of the case where α is a small constant and so it does not fit exactly into the assumptions of Theorem 2. In $K_{s,t}$ the lower bound is much less than $n/2$ and have no lower bounds for the case where the minimum degree significantly exceeds $n/2$.

We can associate a sequence σ_H of length n over the alphabet $\{A, B\}$ with a Hamilton cycle H in $K_{s,t}$. The ith symbol will be an A if and only if the ith vertex of the cycle is in A. Only AB edges are in G and it is not difficult to show by examining σ_H that at most $2ks$ of the edges of H can be of this type. It follows that if we add edges to G with probability p then the expected number of kth powers will be at most $n!p^{k(n-2s)}$. Thus we require $p \geq n^{-1/k(1-2\alpha)}$ or at least $n^{2-1/k(1-2\alpha)}$ random edges. In particular, for $k = 2$ this implies that we need $n^{2-1/(2-4\alpha)}$ random edges, which for small α yields $n^{3/2-O(\alpha)}$. This is quite optimal, since due to Theorem 1 $\omega n^{3/2}$ is the trivial upper bound.

5 Final Remarks

It follows from Theorem 1, a result of Riordan [14] and the embedding theorem in Dudek, Frieze, Ruciński and Šileikis [4] that w.h.p. the random r-regular $G_{n,r}$ contains the kth power of a Hamilton cycle as long as $n^{1-1/k} \ll r \ll n$, $k \geq 2$.

It might be also of some interest to extend Theorem 2 for any $k \geq 3$ and determine how many random edges need to be added to a graph of order n with minimum degree αn in order that it contains the kth power of a Hamilton cycle w.h.p.

6 Acknowledgments

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