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NOTE

ON THE EXISTENCE OF HAMILTONIAN CYCLES IN A CLASS OF RANDOM GRAPHS

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A digraph with n vertices and fixed outdegree m is generated randomly so that each such digraph is equally likely to be chosen. We consider the probability of the existence of a Hamiltonian cycle in the graph obtained by ignoring arc orientation. We show that there exists m ($\leqslant 23$) such that a Hamiltonian cycle exists with probability tending to 1 as n tends to infinity.

1. Introduction

In this paper digraphs do not have loops or repeated arcs.

For a digraph D let GD be the graph obtained by replacing each directed arc (v, w) by an undirected edge $\{v, w\}$.

For positive integers m and n with $m < n$, let $\mathscr{D}(m, n)$ be the set of all vertex-labelled digraphs with n vertices and mn arcs such that each vertex has outdegree m.

Let $M = |\mathscr{D}(m, n)| = \binom{n-1}{m}^n$. We consider the following problem: If D is chosen at random from $\mathscr{D}(m, n)$ so that each such digraph has probability $1/M$ of being chosen, what is the probability that GD has a Hamiltonian cycle?

The main result of this paper is that there exists $m_0 \leqslant 23$ such that $\lim_{n\to\infty} \text{Prob}(GD \text{ is Hamiltonian}) = 1$ if and only if $m \geqslant m_0$.

One motivation for looking at this problem is that when a random graph is chosen by choosing edges independently with the same probability, Hamiltonian cycles appear (in a probabilistic sense) at the same time that the minimum vertex degree reaches 2 (Komlós and Szemerédi [2]). This requires about $\frac{1}{2}n \log n + n \log\log n$, edges, and its is of interest to try and reduce this number by ensuring, in some way, that each vertex has at least a certain degree.

In a previous paper [1] we studied the probable connectivity of these graphs; work on this was stimulated by Walkup's results on random regular bipartite digraphs [5].

2. Main result

Notation. $V_n = \{1, \ldots, n\}$. For $\alpha > 0$, $V(\alpha, n) = \{S \subseteq V_n : |S| \leq \alpha n\}$.

For a digraph D with vertex set V and arc set A, we define, for $S \subseteq V$ $\delta_D^+(S) = \{w \in V - S$: there exists $v \in S$ such that $(v, w) \in A\}$.

For a graph G with vertex set V and edge set E, we define, for $S \subseteq V$ $\delta_G(S) = \{w \in V - S$: there exists $v \in S$ such that $\{v, w\} \in E\}$.

Lemma 2.1 [1]. *If $m \geq 2$ and $C(m, n) = \{D \in \mathscr{D}(m, n) : GD$ is connected$\}$, then*

$$\lim_{n \to \infty} \text{Prob}(D \in C(m, n)) = 1.$$

By $\text{Prob}(D \in C(m, n))$ we mean $|C(m, n)|/|\mathscr{D}(m, n)|$.

Suppose now $P = (v_1, \ldots, v_k)$ is a longest path in a graph $G = (V, E)$. If $t \neq k-1$ and $\{v_k, v_t\} \in E$, then $P' = (v_1, \ldots, v_t, v_k, v_{k-1}, \ldots, v_{t+1})$ is also a longest path of G. If $s \neq t, t+2$ and $\{v_{t+1}, v_s\} \in E$, we can create another longest path P'' using a similar 'flip'.

Keeping v_1 fixed, let $EP(v_1)$ be the set of other endpoints of longest paths formed by doing all possible sequences of flips.

Lemma 2.2 (Pósa [4]). *If $w \in P - EP(v_1)$, then w is adjacent to a vertex of $EP(v_1)$ in G if and only if w is adjacent to some vertex of $EP(v_1)$ on P.*

Corollary 2.3. $|\delta_G(EP(v_1))| \leq 2\,|EP(v_1)| - 1$.

Remark 2.4. We can of course take each $v \in EP(v_1)$ and use this as a fixed endpoint to create another set of endpoints $EP(v)$ satisfying Corollary 2.3.

Lemma 2.5. *Let $A(\alpha, m\, n) = \{D \in \mathscr{D}(m, n)$: $S \in V(\alpha, n)$ implies $|\delta_D^+(S)| \geq 3\,|S|\}$. Then, for any $\alpha < \frac{1}{4}$, there exists $m(\alpha)$ such that, for $m \geq m(\alpha)$,*

$$\text{Prob}(D \in \mathscr{D}(m, n) - A(\alpha, m, n)) = O(1/n).$$

Proof. If $D \in \mathscr{D}(m, n) - A(\alpha, m, n)$, then there exists $S \subseteq V_n$ with $|S| \leq \alpha n$, and $T \subseteq V_n - S$ with $|T| = 3\,|S| - 1$ such that $\delta_D^+(S) \subseteq T$. The probability of this event is clearly bounded above by

$$\sum_{k=\lceil (m+2)/4\rceil}^{\lfloor \alpha n\rfloor} \frac{n!}{k!\,(3k-1)!\,(n-4k+1!)}\left(\binom{4k-2}{m}\Big/\binom{n-1}{m}\right)^k$$

$$\leq \frac{1}{2\pi}\sum_k \frac{3(1+o(1))k}{n-4k+1}\left(\frac{n}{3k^2(n-4k)}\right)^{1/2}\frac{n^n}{k^k(3k)^{3k}(n-4k)^{n-4k}}\left(\frac{4k}{n}\right)^{km}$$

$$=\frac{1}{2\pi}\sum_k \left(\frac{3(1+o(1))n}{(n-4k)^3}\right)^{1/2}\left(\left(\frac{4k}{n}\right)^{m-4}\frac{256}{27}\right)^k\left(\frac{n-4k}{n}\right)^{4k-n}.$$

Let m be such that

$$\varepsilon = m-4-\sup_{0<x\leq\alpha}\left(\frac{(1-4x)\log(1-4x)-x\log(256/27)}{x\log 4x}\right)>0.$$

Then for $1\leq k\leq \alpha n$ we have

$$\left(\left(\frac{4k}{n}\right)^{m-4}\frac{256}{27}\right)^k\left(\frac{n-4k}{n}\right)^{4k-n}\leq\left(\frac{4k}{n}\right)^{\varepsilon k}.$$

Thus the probability in question is bounded above by

$$\frac{1}{2\pi}\sum_{k=1}^{\alpha n}\left(\frac{3(1+o(1))n}{(n-4k)^3}\right)^{1/2}\left(\frac{4k}{n}\right)^{\varepsilon k}=O(n^{-(1+\varepsilon)}).\quad\square$$

Definition. A graph G has *property LC* if a longest cycle of G has the same number of vertices as a longest path of G.

Lemma 2.6. *Let $B=B(m,n)=\{D\in\mathcal{D}(m,n)$: GD does not have Property LC$\}$. For $\alpha>0$ and $m\geq m(\alpha)$,*

$$\text{Prob}(D\in B)\leq((m/(m-2))^{1/2}(1-\alpha)^{\alpha})^n+O(1/n).$$

Proof. Let $\mathcal{D}(m,n)=\{D_1,\ldots,D_M\}$ and suppose that $A=A(\alpha,m,n)=\{D_1,\ldots,D_{M'}\}$. It follows from Lemma 2.5 that $1-M'/M=O(1/n)$.

Given $D_i\in A$, construct $N=m^n$ coloured digraphs $D_{i1},\ldots,D_{iN}$ as follows: for each vertex v of D_i choose one arc leaving v and colour it green; colour the remaining arcs blue.

We note that, for $D_i\in A$, the blue subdigraph Δ_{ij} of D_{ij} satisfies

$$S\in V(\alpha,n)\quad\text{implies}\quad|\delta_\Delta^+(S)|\geq 2\,|S|,\quad\text{where } \Delta=\Delta_{ij}. \tag{2.1}$$

Next let $a_{ij}=1$ if no arc joins 2 endpoints of a longest path of $G\Delta_{ij}$; otherwise $a_{ij}=0$.

We show next that if $D_i\in B$, then

$$\sum_{j=1}^{N} a_{ij}\geq N_1=((m-2)/m)^{n/2}N. \tag{2.2}$$

Suppose then that $D_i \in B$ and GD_i has a longest path P with k vertices. At least one colouring t yields a GD_{it} in which the edges of P are blue. Clearly $a_{it} = 1$ since $D_i \in B$.

Now in D_{it} fix $k-1$ arcs Q that together produce P (there may be some choice here). In the subdigraph induced by these $k-1$ arcs, let there be k_i vertices of outdegree $i = 0, 1, 2$.

Now, for any colouring j of D_i in which the arcs in Q are coloured blue, $a_{ij} = 1$. Since there are

$$m^{n-k} \prod_{i=0}^{2} (m-i)^{k_i} \geq m^{n-k/2}(m-2)^{k/2}$$

such colourings, (2.2) follows immediately.

Thus, if $M_1 = |A \cap B|$,

$$M_1 \leq \sum_{i=1}^{M'} \sum_{j=1}^{N} a_{ij}/N_1 \tag{2.3}$$

To bound the double sum we construct the following partition: for $\Delta \in \mathscr{D}(m-1, n)$, let $X_\Delta = \{D_{ij} : \Delta_{ij} = \Delta\}$. Let $N_\Delta = |\{D_{ij} \in X_\Delta : i \leq M', a_{ij} = 1\}|$. We shall show next that

$$N_\Delta \leq (1-\alpha)^{\alpha n}(n-m)^n \quad \text{for all } \Delta. \tag{2.4}$$

Thus

$$\sum_{i=1}^{M'} \sum_{j=1}^{N} a_{ij} = \sum_{\Delta \in \mathscr{D}(m-1,n)} N_\Delta$$
$$\leq (1-\alpha)^{\alpha n}(n-m)^n \, |\mathscr{D}(m-1, n)| = (1-\alpha)^{\alpha n} MN.$$

Then, from (2.2) and (2.3), $M_1 \leq (1-\alpha)^{\alpha n}(m/(m-2))^{n/2}M$. The result now follows as $|B| \leq M_1 + (M - M')$.

To prove (2.4), select a particular Δ. Let $P = (v_1, \ldots, v_k)$ be some longest path of $G\Delta$. Let $EP = EP(v_1) \cup \{v_1\}$ and for $v \in EP$ let $EP(v)$ be defined as in Remark 2.4.

It follows from Corollary 2.3 and (2.1) that, if $N_\Delta > 0$, $|EP| \geq \alpha n$ and $|EP(v)| \geq \alpha n$ for all $v \in EP$.

Now consider all ways of adding 1 new green arc to each vertex of Δ. There are $(n-m)^n$ ways of doing this and of these no more than $(1-\alpha)^{\alpha n}(n-m)^n$ ways avoid joining some $v \in EP$ to some $v \in EP(v)$; this is necessary if the coloured digraph constructed is to be a D_{ij} with $a_{ij} = 1$. (2.4) now follows. □

Using the above lemma, we are now able to prove the main result.

Theorem 2.7. *Let $H(m, n) = \{D \in \mathscr{D}(m, n) : GD$ has a Hamiltonian cycle$\}$. There exists m_0 such that, for $m \geq m_0$,*

$$\lim_{n \to \infty} \text{Prob}(D \in H(m, n)) = 1.$$

Proof. Fix $\alpha > 0$ and $m_0 \geq m(\alpha)$ such that $1 - 2/m_0 > (1-\alpha)^{2\alpha}$ and choose $D \in \mathscr{D}(m_0, n)$ at random. Suppose that a longest path in GD has k vertices. We know from Lemma 2.6 that, with probability tending to 1, GD has a cycle C with k vertices. Now, if $k < n$, GD is not connected as no vertex of C can be joined to a vertex not in C, since GD has no path with $k+1$ vertices. But for $m \geq 2$, by Lemma 2.1, the probability that GD is connected tends to 1. Thus $k = n$ with probability tending to 1 and the result follows. □

Remark 2.8. We know that $m_0 \leq 23$ by taking $\alpha = 0.202$ in Lemmas 2.5 and 2.6.

We conjecture that the smallest value of m_0 is 3.

Remark 2.9. The problem of when D (as opposed to GD) has a Hamiltonian cycle has been solved by McDiarmid [3]; $m = \log\ n$ is (about) the required value for m.

Using a similar technique, we have recently proved [6] that, for $r \geq 796$, the probability that a random vertex-labelled r-regular graph with n vertices is Hamiltonian tends to 1 as n tends to infinity. Bollobás [7] has obtained a similar result.

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