Name: ________________________________

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Q1: (40pts)
Suppose that $p = d/n$ where $d$ is constant. Prove that w.h.p., in $G_{n,p}$, no vertex belongs to more than one triangle.

**Solution:** If there is a vertex $v$ that lies in more than one triangle, then there is a set of $k = 4, 5$ vertices that contain at least $k + 1$ edges. The probability of this can be bounded by

$$\sum_{k=4}^{5} \binom{n}{k} 2^k \left( \frac{d}{n} \right)^{k+1} \leq \frac{1}{n} \sum_{k=4}^{5} \frac{2^k d^{k+1}}{k!} = o(1).$$
Q2: (30pts)
Suppose that $c \neq 1$ is constant and that $\epsilon_n = 1/\log \log n$. Show that w.h.p. the length of the longest path component in $G_{n,p}$, $p = \frac{\epsilon_n}{n}$ is $(1 \pm \epsilon_n) \frac{\log n}{c - \log c}$.

Solution: Let $L_{\pm} = (1 \pm \epsilon_n) \frac{\log n}{c - \log c}$ and let $X_{-}$ denote the number of components that are paths of length $L_{-}$ and let $X_{+}$ denote the number of components that are paths of length at least $L_{+}$.

Now we know that w.h.p. there are no path components of length more than $A \log n$ for some constant $A > 0$. This is because (a) there are no components of size greater than $A \log n$ and (b) the giant component has too many edges to be a path. Thus,

$$
P(X_{+} > 0) \leq o(1) + \sum_{k=L_{+}+1}^{A \log n} \binom{n}{k} \frac{1}{k!} \left( \frac{c}{n} \right)^{k-1} \left( 1 - \frac{c}{n} \right)^{k(n-k)}
$$

$$
\leq o(1) + \frac{2A \log n}{c} \sum_{k=L_{+}+1}^{A \log n} (ce^{-c})^k
$$

$$
\leq o(1) + \frac{2A \log n}{c} \cdot (ce^{-c})^{L_{+}}
$$

$$
\leq o(1) + \frac{2A \log n}{c} \cdot \frac{(ce^{-c})^{\epsilon_n L_{+}}}{n}
$$

$$
= o(1).
$$

Now, with $X = X_{-}$ and $L = L_{-}$,

$$
E(X) = \left( \frac{n}{L+1} \right) (L+1)! \left( \frac{c}{n} \right)^L \left( 1 - \frac{c}{n} \right)^{L(n-L)-(\frac{L+1}{2})+L}
$$

$$
\geq (1 - o(1)) n (ce^{-c})^L
$$

$$
= (1 - o(1)) (ce^{-c})^{-\epsilon_n L}
$$

$$
\rightarrow \infty.
$$

Furthermore, if $P_1, P_2, \ldots, P_M$ is an enumeration of the paths of length $L$ in $K_n$ and $X_i$ is the indicator for $P_i$ being a path component then

$$
E(X^2) = E(X) + \sum_{i \neq j} P(X_i = X_j = 1)
$$

$$
\leq E(X) + E(X)^2.
$$
This is because if $i \neq j$ then
\[
\mathbf{P}(X_i = X_j = 1) = \begin{cases} 
\mathbf{P}(X_i = 1)\mathbf{P}(X_j = 1) & P_i \cap P_j = \emptyset, \\
0 & P_i \cap P_j \neq \emptyset.
\end{cases}
\]
Thus
\[
\mathbf{P}X > 0 \geq \frac{\mathbf{E}(X)^2}{\mathbf{E}(X^2)} \geq \frac{1}{\mathbf{E}(X)} + 1 \to 1.
\]
Q3: (30pts)
Let \( C \) denote the set of connected unicyclic graphs on vertex set \([n]\). Suppose that \( Z \) is the length of the unique cycle \( C_H \) in a randomly chosen member \( H \in C \). Show that, where \( N = \binom{n}{2} \),

\[
E[Z] = \frac{n^{n-2}(N - n + 1)}{|C|}.
\]

Hints: Count the number \( X \) of pairs \((H, e)\) in two ways, where \( e \in C_H \) and \( H \in C \). Let \( X_k \) denote the number of \( H \in C \) with \(|C_H| = k\).

Solution: First,

\[
X = \sum_{\text{spanning trees } T} |\{ e \notin E(T) \}| = n^{n-2}(N - n + 1).
\]

This is because there are \( n^{n-2} \) spanning trees and each \( e \notin E(T) \) creates a cycle when added to \( T \).

On the other hand,

\[
X = \sum_{k=1}^{n} kX_k
\]

since each \( H \) with \(|C_H| = k\) gives rise to \( k \) pairs, by deleting an edge of the unique cycle. Thus,

\[
|C|E(Z) = |C| \sum_{k=1}^{n} k \frac{X_k}{|C|} = n^{n-2}(N - n + 1).
\]