1. A tournament $T$ is an orientation of the complete graph $K_n$. In a random tournament, edge $\{u, v\}$ is oriented from $u$ to $v$ with probability $1/2$ and from $v$ to $u$ with probability $1/2$. Show that w.h.p. a random tournament is strongly connected.

**Solution:** If $T$ is not strongly connected then there exists a set $S$ of size at most $n/2$ such that all edges in $S : \bar{S}$ are oriented the same way i.e all are $S$ to $\bar{S}$ or vice-versa. The probability of this is at most

$$2 \sum_{k=1}^{n/2} \binom{n}{k} \frac{1}{2^{k(n-k)}} \leq 2 \sum_{k=1}^{n/2} \left( \frac{ne}{k2^{n-k}} \right)^k = o(1).$$

2. Let $T$ be a random tournament. Show that w.h.p. the size of the largest acyclic sub-tournament is asymptotic to $2 \log_2 n$. (A tournament is acyclic if it contains no directed cycles).

**Solution:** Let $X_k$ denote the number of sets of size $k$ that induce an acyclic tournament. If $S$ is acyclic then $S$ can be ordered $x_1, x_2, \ldots, x_k$ so that if $i < j$ then the edge is oriented from $x_i$ to $x_j$. Thus,

$$\mathbb{E}(X_k) \leq \binom{n}{k} k! \frac{1}{2^{k(k-1)/2}} \leq \left( \frac{ne}{2^{(k-1)/2}} \right)^k.$$

So, $\mathbb{E}(X_k) \to 0$ if $k \geq (2 + \varepsilon) \log_2 n$. If $k \leq (2 - \varepsilon) \log_2 n$ then the second moment method suffices.

3. Suppose that the edges of $G_{n,p}$ where $0 < p \leq 1$ is a constant, are given exponentially distributed weights with rate 1. Show that if $X_{ij}$ is the shortest distance from $i$ to $j$ then

(a) For any fixed $i, j$,

$$\mathbb{P} \left( \left| \frac{X_{ij}}{\log n/n} - \frac{1}{p} \right| \geq \varepsilon \right) \to 0.$$

(b) $$\mathbb{P} \left( \left| \max_j X_{ij} \frac{1}{\log n/n} - \frac{2}{p} \right| \geq \varepsilon \right) \to 0.$$

**Solution:** one argues that the number of edges between any set $S$ of size $k$ and its complement $\bar{S}$ is $(1 + o(1))k(n - k)p$. This follows from the Chernoff bounds. It follows that the expression for $\mathbb{E}(Y_n)$ in Chapter 19.2 of the book becomes $\mathbb{E}(Y_n) \approx \sum_{k=1}^{n-1} \frac{1}{k(n-k)p}$. The rest of the proof of this section is only changed by the factor $1/p$. 

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