Homework 8: Solutions

12.3.10 Suppose that $r \geq (1 + \epsilon)r_0$, as in Theorem 11.8. Show that if $1 \leq k = O(1)$ then $G_{X,r}$ is $k$-connected w.h.p.

Solution: Let $T$ be a spanning tree of $\Gamma$ as promised by Lemma 11.11 and let $T^*$ denote the vertices of $X$ in the cells of $T$. If we remove $O(1)$ vertices from $G$ then what remains of $T^*$ will be connected. This is because each vertex of $T$ corresponds to a clique of size $\Omega(\log n)$. Removing vertices from bad cells cannot disconnect the graph, as (11.16) tells us that we connect the vertices of bad cells, directly to $T^*$.

12.3.11 Show that if $2 \leq k = O(1)$ and $r \gg \sqrt{\log n}$ then w.h.p. $G_{X,r}$ contains a $k$-clique. On the other hand, show that if $r = o(\log n)$ then $G_{X,r}$ contains no $k$-clique.

Solution: Let $Z$ denote the number of $k$-cliques. Suppose first that $r = \omega n^{2(k-1)/k}$ where $\omega \to \infty$. Then

$$E(Z) \leq n \left( \frac{n}{k-1} \right)^{(\pi r^2)^{k-1}} \leq n^k \left( \frac{\pi}{\omega^2 n^{2(k-1)/k}} \right)^{k-1} = \frac{\pi^{k-1}}{\omega^{2(k-1)}} = o(1).$$

Suppose now that $r = \omega n^{2(k-1)/k}$ where $\omega \to \infty$, $w = o(\log n)$. We there are exactly $k-1$ points within distance $r/2$ of $x$ and the remaining $n-k-o(n)$ points are at distance $2r$ or more from $x$. Let $Z = |\{x : E_x \text{ occurs}\}|$.

$$E(Z) \geq n \left( \frac{n-o(n)}{k-1} \right)^{(\pi r^2)^{k-1}} \left( 1 - 4\pi r^2 \right)^{n-k-o(n)} \geq n^k \left( \frac{\pi \omega^2}{4n^{2(k-1)/k}} \right)^{k-1} = \frac{\pi^{k-1} \omega^{2(k-1)}}{2k^2 \pi^{k-1}} \to \infty.$$

We use the Chebyshev inequality. We have $E(Z^2) \leq E(Z)^2$. This is because if $E_x, E_y$ both occur then $|x-y| \geq 2r$ and the expectation of $Z^2$ simplifies.

12.3.12 Suppose that $r \gg \sqrt{\frac{\log n}{n}} = o(1)$. Show that w.h.p. the diameter of $G_{X,r} = \Theta \left( \frac{1}{r} \right)$.

Solution: We partition $D = [0,1]^2$ into $m^2$ cells of size $1/m \times 1/m$ where $m = \lceil 10/r \rceil$. Because $r \gg \sqrt{\frac{\log n}{n}}$ we know that w.h.p. each cell is non-empty. Furthermore, if $X,Y \in X'$ lie in adjacent cells then they are connected by an edge in $G = G_{X,r}$. It follows that we can connect pairs of vertices in $G$ by a path of length at most $2m$. We simply follow a sequence of adjacent cells. Thus the diameter of $G$ is $O(1/r)$. The diameter is clearly $\Omega(1/r)$ because any path from a vertex in the top left-most cell to a vertex in the top right-most cell must use at least $\frac{1-2/m}{r}$ edges.