Let $m_0 = n(\log n - \log \log n)$ and $m_1 = n(\log n + \log \log n)$. We know from Theorem 6.1 that w.h.p. the hitting time $m_{pn}$ for having a perfect matching satisfies $m_0 \leq m_{pn} \leq m_1$. Now w.h.p. at time $m_0$ there does not exist $S \subseteq A, 2 \leq |S| \leq n - 1$ such that $|N(S)| < |S|$. This follows from the calculations in Theorem 6.1. Clearly this remains true for all $m \in [m_0, m_1]$. Thus w.h.p. the only witnesses are singletons or the set $A$ and the singletons will cease to be witnesses when they are no longer isolated and $A$ will cease to be a witness once there are no isolated vertices in $B$.

Let $m = Np$ and let $m = m_1 + m_2$ where $m_2 = \omega n/2$. First one can show that in $G_1 = G_{n,n,m_1}$:

1. W.h.p. $S \subseteq A, 1 \leq |S| \leq \epsilon_k n$ implies that $|N(S)| \geq k|S|$ where $\epsilon_k$ depends only on the constant $k$.

2. Let $G$ be a bipartite graph with vertex partition $X,Y, |X| = |Y|$ that does not contain a perfect matching. Let $M$ denote the set of maximum size matchings in $G$. Suppose that $M \in M$ is a maximum matching in $G$ and $v \in X$ is not covered by $M$ and

$$A(v) = \{ w \in B : \exists M' \in M \text{ s.t. } v,w \text{ are not covered by } M' \}.$$ 

Then, $|N(A(v))| < |A(v)|$.

Property 1. follows the lines of the proof of Lemma 6.4 and Property 2. follows the lines of Lemma 6.3. Also, Property 2. can be applied to each $w \in A(v)$ to construct for each $w$, a set $B(w)$ of vertices for which each $z \in B(w)$ there is a matching $M \in M$ that isolates $w,z$. Note that adding the edge $(z,w)$ will increase the size of a maximum matching. We call such an edge a **booster**.

Now let $E(G_{n,m}) = \{ e_1, e_2, \ldots, e_m \}$ and let $E_1 = \{ e_1, e_2, \ldots, e_{m_1} \}$ and $E_2 = \{ e_{m_1+1}, e_{m_1+2}, \ldots, e_m \}$. Suppose that $G_1$ does not contain $k$ edge-disjoint perfect matchings and that it contains $\ell < k$ edge-disjoint perfect matchings $M_1, M_2, \ldots, M_\ell$ and that $\mu$ denotes the size of the largest matching in $G_1$ left after removing $\ell$ edge-disjoint perfect matchings.

It follows from Properties 1. and 2. that there is a set of size at least $c_k^2 n^2$ boosters. Indeed, $S = \{ (z,w) : w \in A(v), z \in B(w) \}$ suffices, since Property 1. implies that w.h.p. after removing $M_1, M_2, \ldots, M_\ell$ we have $S \subseteq A, 1 \leq |S| \leq \epsilon_k n$ implies that $|N(S)| \geq |S|$ and Property 2. implies that the $A(v), B(w) \in A(v)$ violate the neighborhood property.

Now consider adding the edges $E_2$ one by one to $G_1$. Each such edge has probability $\geq c_k^2$ of being a booster. After we have found at most $kn$ boosters, we will have $k$ edge disjoint perfect matchings.
So, the probability that $G_{n,m}$ fails to have $k$ edge disjoint perfect matchings can be bounded by

$$o(1) + P(Bin(\omega n/2, \epsilon_k^2) \leq kn) = o(1).$$

6.7.8 Following the hint we partition $[n]$ into 3 sets $A,B,C$ of size $n/3$. The bipartite graph $H$ induced by $A,B$ is distributed as $G_{n/3,n/3,p}$ and since $\frac{n}{3}p \gg \log \frac{n}{3}$ this graph has a perfect matching w.h.p. Fix a perfect matching $M$ of $H$ and define another random bipartite graph $K$ with vertices $M,C$ and an edge $(e,x)$ for each $e = \{u,v\} \in M, x \in C$ such that the edges $\{x,u\}, \{x,v\}$ both exist. The random graph $K$ is distributed as $G_{n/3,n/3,p^2}$ and since $\frac{n}{3}p^2 \gg \log \frac{n}{3}$ this graph has a perfect matching w.h.p. This perfect matching corresponds to $n/3$ vertex disjoint triangles.