3.3.2 Let $Z_k$ denote the number of vertices of degree $k$ in the giant component of $G_{n,p}$. Fix a vertex, say vertex $n$ and consider $H_1 = G_{n-1,p}$. We know that w.h.p. $H_1$ will have a giant component $C$ of size $\approx (1 - \frac{c}{k}) n$. Then, $n$ will be part of the giant component of $G_{n,p}$ and have degree $k$ if (i) it has degree $k$ and (ii) at least one of its neighbors is in $C$. Let $p_k$ denote the probability of (i), (ii). (This assumes that $H_1$ has a giant and this fails to happen with probability $o(1) = o(p_k)$. Now the probability of (i) is asymptotically equal to $c^{k-\epsilon} k!$ and given (i), the neighbors of $n$ will be a uniform random $k$-subset of $[n-1]$. And so the probability (ii) fails to hold, given (i) is asymptotically equal to $(\frac{c}{k})^k$. Thus $p_k$ is asymptotically equal to $c^{k-\epsilon} (1 - (\frac{c}{k})^k)$. This shows that $E(Z_k) \approx np_k$.

To show $Z_k \approx np_k$ w.h.p. we can use the second moment method. For this we consider $H_2 = G_{n-2,p}$ and the probability that both $n, n-1$ are degree $k$ vertices of the giant. This is at most $p + (1 + o(1))p^2_0$, where $p$ accounts for $n, n-1$ being adjacent. Thus $\text{Var}(Z_k) \approx E(Z_k)^2$ and the Chebyshev inequality finishes the proof.

4.3.1 Now we know that $m^*_+ \in [m_- , m_+]$. It follows that the expected degree of $v$ in $G_{m_+}$ is $O(\log \log n)$ and then the Markov inequality implies that the degree of $v$ in $G_{m_+}$ is $\leq (\log \log n)^2$ w.h.p.

We now consider the probability $p_0$ that $G_{m_+}$ contains 4 vertices $w, x, y, z$ such that (i) the degree of $w$ is at most $(\log \log n)^2$ and (ii) $x, y, z$ form a triangle and (iii) $w$ is adjacent to $x$. Let $p_+ = \frac{m^*_+}{n^2}$ then Lemma 1.3 implies that $P((ii), (iii) \text{ in } G_{m_+}) \leq 3p^4_+$. Condition on (ii), (iii). Then if $L = (\log \log n)^2$ we have, on using Lemma 1.3 once more,

$$P((i) | (ii), (iii)) \leq 3 \sum_{k=0}^{L} \binom{n-4}{k} p^k_+ (1 - p_+)^{n-4-k} = \frac{1}{n^{1-o(1)}}.$$ 

It follows that $P(\exists w, x, y, z) \leq n^4 \times 3p^4_+ \times \frac{1}{n^{1-o(1)}} = o(1)$.

4.3.3 Let $p = \frac{m}{n} = \frac{d}{n-1} = \frac{\log n}{n^2}$. Monotonicity implies that it suffices to consider $G_{n,p}$. Now observe that $|S_{i+1}|$ is distributed as the binomial $\text{Bin}(n-|S_i|, 1 - (1 - p)^{|S_i|})$ where $S_{\leq i} = \bigcup_{j \leq i} S_j$. Assume inductively that the event $B_i$ holds where $B_i = \bigcap_{j \leq i} A_j$ and $A_j = \left\{ \left( \frac{d}{2} \right)^j \leq |S_j| \leq \left( \frac{5d}{4} \right)^j \right\}$.
We show that if $i_0 = \left\lceil \frac{2 \log n}{3 \log d} \right\rceil$ then $P(B_{i_0}) = 1 - o(1)$. Now using the Chenoff bounds, we see that

$$P(\|S_i\| - d \geq \epsilon d) \leq 2e^{-\epsilon^2 d/3} \leq n^{-1}$$

if $\epsilon = \omega^{-1/2}$. And so we have the basis of our induction. Observe next that if $B_i, i \geq 1$ occurs then $\omega \log n \leq |S_i| \leq n^{59/100}$.

**Case 1:** $d \leq n^{501/1000}$.
In this case $n^{99/100}p^2 = o(1)$ and so

$$1 - (1 - p)^{|S_i|} \approx |S_i|p.$$ 

Also, $n - |S_i| = n - o(n)$ and so

$$E(\text{Bin}(n - |S_{\leq i}|, 1 - (1 - p)^{|S_i|})) \approx |S_i|np \geq \frac{1}{2} \omega |S_i| \log n.$$ 

Applying the Chernoff bounds, we see that

$$P\left( \|S_{i+1}\| - d|S_i| \geq \frac{d}{100} |S_i| \left| B_i \right\right) \leq n^{-1}.$$ 

This completes the induction.

**Case 2:** $d \geq n^{501/1000}$.
In this case $G_{n,m}$ has minimum degree $\approx d$ and diameter two and so there is nothing to prove.