Homework 2: Solutions

2.4.10 We know from Theorem 2.14 that the giant component has size \( \approx \left(1 - \frac{x}{1+\epsilon}\right)n \) where \( xe^{-x} = (1 + \epsilon)e^{-1-\epsilon} \). Suppose then \( x = 1 - y \) then

\[
(1 - y)e^{-1+y} = (1 + \epsilon)e^{-1-\epsilon}
\]
or after multiplying both sides by \( e \),

\[
(1 - y) \left(1 + y + \frac{y^2}{2} + O(y^3)\right) = (1 - \epsilon) \left(1 - \epsilon + \frac{\epsilon^2}{2} + O(\epsilon^3)\right)
\]
or

\[
1 - \frac{y^2}{2} + O(y^3) = 1 - \frac{\epsilon^2}{2} + O(\epsilon^3)
\]
or

\[
|y - \epsilon| = O \left(\frac{\epsilon^3}{y + \epsilon}\right) = O(\epsilon^2).
\]

This implies that \( y - \epsilon = O(\epsilon^2) \) and then the size of the giant is

\[
\left(1 - \frac{1 - \epsilon + O(\epsilon^2)}{1 + \epsilon}\right)n = (2\epsilon + O(\epsilon^2))n.
\]

2.4.14 The expected number of sets of size at most \( s \) that contain at least \( ks/2 \) edges is at most

\[
\sum_{t=2k+1}^{s} \binom{n}{t} \left(\frac{l}{kt/2}\right)^{p^{kt/2}} \leq \sum_{t=2k+1}^{s} \binom{ne}{t} \left(\frac{t^2e}{kt}\right)^{kt/2} p^{kt/2}
\]

\[
= \sum_{t=2k+1}^{s} \left(\frac{t}{n}\right)^{k/2-1} \left(\frac{e^{1+2/kC}}{k}\right)^{k/2} t = o(1)
\]

if say, \( s \leq s_0 = \theta n \) where \( \theta = \frac{1}{2}(e^{1+2/kC/k})^{-k/(k-2)}n \).

This means that w.h.p. every set of size at most \( s_0 \) contains a vertex with fewer than than \( k \) neighbors in the set. Thus w.h.p. either the \( k \)-core is empty or it has size greater than \( s_0 \).