Separation of largest degrees, Graph isomorphism and edge coloring.

Lemma

Let $k = (n-1)p + x\sqrt{(n-1)pq}$, p constant, $q = 1-p$, where $x \le (\log n)^2$ (for convenience).

Then

$$B_k = \binom{n-1}{k} p^k (1-p)^{n-1-k} = (1+o(1))\sqrt{\frac{1}{2\pi npq}}\, e^{-x^2/2}.$$

Proof

Stirling's Formula gives

$$B_k = (1+o(1))\sqrt{\frac{1}{2\pi npq}} \left(\left(\frac{(n-1)p}{k}\right)^{\frac{k}{n-1}} \left(\frac{(n-1)q}{n-1-k}\right)^{1-\frac{k}{n-1}} \right)^{n-1}.$$

Now

$$\left(\frac{k}{(n-1)p}\right)^{\frac{k}{n-1}} = \left(1 + x\sqrt{\frac{q}{p(n-1)}}\right)^{k/n-1}$$

$$= \exp\left\{\left(x\sqrt{\frac{q}{p(n-1)}} - \frac{x^2}{2}\frac{q}{p(n-1)} + O\left(n^{-3/2}\right)\right)\left(p + x\sqrt{\frac{pq}{n-1}}\right)\right\}$$

$$= \exp\left\{x\sqrt{\frac{pq}{n-1}} + \frac{x^3}{2}\cdot\frac{q}{n-1} + O\left(n^{-3/2}\right)\right\}$$

$$\left(\frac{n-1-k}{(n-1)q}\right)^{1-k/n-1} = \left(1 - x\sqrt{\frac{p}{q(n-1)}}\right)^{1-k/n-1}$$

$$= \exp\left\{ -\left(x\sqrt{\frac{p}{q(n-1)}} + \frac{x^2}{2}\cdot\frac{p}{q(n-1)} + O(n^{-3/2}) \right)\left(q - x\sqrt{\frac{pq}{n-1}} \right) \right\}$$

$$= \exp\left\{ -x\sqrt{\frac{pq}{n-1}} + \frac{x^2}{2}\frac{p}{n-1} + O(n^{-3/2}) \right\}.$$

So

$$\left(\frac{k}{(n-1)p}\right)^{k/n-1}\left(\frac{n-1-k}{(n-1)q}\right)^{1-k/n-1} =$$

$$\exp\left\{\frac{x^2}{2(n-1)} + O(n^{-3/2})\right\}$$

Substituting into

$$(1+o(1))\sqrt{\frac{1}{2\pi npq}}\left(\left(\frac{(n-1)p}{k}\right)^{\frac{k}{n-1}}\left(\frac{(n-1)q}{n-1-k}\right)^{1-\frac{k}{n-1}}\right)^{n-1}$$

gives required expression. $\square$

Lemma

Let $\epsilon = \frac{1}{10}$ and p be constant

$$k_{\pm} = (n-1)p + (1 \pm \epsilon)\sqrt{2(n-1)pq\log n}.$$

Then whp

(i) $\Delta(G_{n,p}) \leq k_+$

(ii) There are $\Omega(n^{2\epsilon(1-\epsilon)})$ vertices of degree at least k_-

(iii) $\nexists$ $u \neq v$ such that $d(u), d(v) \geq k_-$ and $|d(u) - d(v)| \leq 10$.

We first prove that as $x \to \infty$

$$\frac{1}{x} e^{-x^2/2}\left(1-\frac{1}{x^2}\right) \le \int_x^\infty e^{-y^2/2}\,dy \le \frac{1}{x} e^{-x^2/2}. \qquad (***)$$

<u>Proof</u>

$$\int_x^\infty e^{-y^2/2}\,dy = -\int_x^\infty \frac{1}{y}\left(e^{-y^2/2}\right)'\,dy$$

$$= -\left[\frac{1}{y} e^{-y^2/2}\right]_x^\infty - \int_x^\infty \frac{1}{y^2} e^{-y^2/2}\,dy$$

$$= \frac{1}{x} e^{-x^2/2} + \left[\frac{1}{y^3} e^{-y^2/2}\right]_x^\infty + 3\int_x^\infty \frac{1}{y^4} e^{-y^2/2}\,dy$$

$\square$

(i) Let X be the number of vertices of degree k.

$$E(X_k) = (1+o(1))\sqrt{\frac{n}{2\pi pq}}\ \exp\left\{-\frac{1}{2}\left(\frac{k-(n-1)p}{\sqrt{(n-1)\ pq}}\right)^2\right\}$$

assuming that $k \le k_2 = (n-1)p + (\log n)^2\sqrt{(n-1)pq}$.

But if $k > k_2$ then

$E(X_k) \le E(X_{k_2})$ – binomial $\searrow$ after mean

$\approx n\exp\{-\Omega((\log n)^4)\}$

$= o(1)$.

So, $Y_k = X_k + X_{k+1} + \cdots$

$$E(Y_k) \approx \sum_{l=k}^{k_L} \sqrt{\frac{n}{2\pi p q}} \exp\left\{-\frac{1}{2}\left(\frac{l-(n-1)p}{\sqrt{(n-1)\,p\,q}}\right)^2\right\}$$

$$\approx \sum_{l=k}^{\infty} \sqrt{\frac{n}{2\pi p q}} \exp\left\{-\frac{1}{2}\left(\frac{l-(n-1)p}{\sqrt{(n-1)\,p\,q}}\right)^2\right\}$$

$$\approx \sqrt{\frac{n}{2\pi p q}} \int_{\lambda=k}^{\infty} \exp\left\{-\frac{1}{2}\left(\frac{\lambda-(n-1)p}{\sqrt{(n-1)pq}}\right)^2\right\} d\lambda$$

If $k = (n-1)p + x\sqrt{(n-1)pq}$ then

$$\sqrt{\frac{n}{2\pi pq}} \int_{\lambda=k}^{\infty} \exp\left\{-\frac{1}{2}\left(\frac{\lambda-(n-1)p}{\sqrt{(n-1)pq}}\right)^2\right\} d\lambda$$

$$= \sqrt{\frac{n}{2\pi pq}} \cdot \sqrt{(n-1)pq} \cdot \int_{y=x}^{\infty} e^{-y^2/2}\, dy$$

$$\simeq \frac{n}{\sqrt{2\pi}} \cdot \frac{1}{x} \cdot e^{-x^2/2}$$

When $k = k_+$, $x = (1+\epsilon)\sqrt{2\log n}$ and (1) follows.

When $k = k_*$, $x = (1-\epsilon)\sqrt{2\log n}$

and $$E(Y_k) = \Omega\left(n^{2\epsilon(1-\epsilon)}\right) \to \infty.$$

We use the second moment method to show concentration.

$$E(Y_k(Y_{k-1}-1)) = n(n-1)\sum_{k \le k_1, k_2}^{\le k_*} \Pr(d(1) = k_1 \wedge d(2) = k_2)$$

$$= n(n-1)\left[\sum_{k_1, k_2} p\left(\hat{d}(1) = k_1 - 1 \wedge \hat{d}(2) = k_2 - 1\right) + (1-p)P\left(\hat{d}(1) = k_1 \wedge \hat{d}(2) = k_2\right)\right]$$

where $\hat{d}$ = # nbrs in $\{3, 4, \ldots, n\}$.

$$= n(n-1) \sum_{k_1, k_2} \Big[p P(\hat{d}(1) = k_1 - 1) P(\hat{d}(2) = k_2 - 1) + (1-p) P(\hat{d}(1) = k_1) P(\hat{d}(2) = k_2) \Big]$$

$$\frac{P(\hat{d}(1) = k_1 - 1)}{P(\hat{d}(1) = k_1)} = \frac{\binom{n-2}{k_1 - 1}(1-p)}{\binom{n-2}{k_1} p} = \frac{k_1 (1-p)}{(n-2-k_1) p} = 1 + \tilde{O}(n^{-1/2}).$$

$$= n(n-1) \sum_{k_1, k_2} \Big[P(\hat{d}(1) = k_1) P(\hat{d}(2) = k_2) \big(1 + \tilde{O}(n^{-1/2})\big) \Big]$$

$$= n(n-1) \sum_{k_1, k_2} \left[P(\hat{d}(1) = k_1) P(\hat{d}(2) = k_2) (1 + \tilde{O}(n^{-1/2})) \right]$$

$$\frac{P(\hat{d}(1) = k_1)}{P(d(1) = k_1)} = \frac{\binom{n-2}{k_1}}{\binom{n}{k_1}} (1-p)^{-2} = 1 + \tilde{O}(n^{1/2})$$

$$= n(n-1) \sum_{k_1, k_2} \left[P(d(1) = k_1) P(d(2) = k_2) (1 + \tilde{O}(n^{-1/2})) \right]$$

$$= E(Y_k)(E(Y_k) - 1)(1 + \tilde{O}(n^{-1/2}))$$

So, with $k = k_-$,

$$\Pr\left(Y_k \le \tfrac{1}{2} E(Y_k)\right)$$

$$\le \frac{E(Y_k(Y_k - 1)) + E(Y_k) - E(Y_k)^2}{E(Y_k)^2/4}$$

$$= \tilde{O}\left(\frac{1}{n^{2\epsilon(1-\epsilon)}}\right)$$

$$= o(1).$$

This completes the proof of the second part.

$$\Pr(\neg(iii)) \leq o(1) + \binom{n}{2} \sum_{k_1 = k_-}^{k_+} \sum_{|k_2 - k_1| \leq 10} \Pr(d(1) = k_1 \wedge d(2) = k_2)$$

$$= o(1) + \binom{n}{2} \sum_{k_1, k_2} \Big[p\, P(\hat{d}(1) = k_1 - 1) P(\hat{d}(2) = k_2 - 1) + (1-p) P(\hat{d}(1) = k_1) P(\hat{d}(2) = k_2) \Big]$$

Now

$$\sum_{k_1, k_2} P(\hat{d}(1) = k_1 - 1) P(\hat{d}(2) = k_2 - 1)$$

$$\leq 21 (1 + \tilde{O}(n^{-1/2})) \sum_{k_1} \Pr(\hat{d}(1) = k_1 - 1)^2$$

and

$$\sum_{k_1} \Pr\left(\hat{d}(1) = k_1 - 1\right)^2 \lesssim \frac{1}{2\pi p q n} \int_{y=x}^{\infty} e^{-y^2}\, dy,$$

where $x = \dfrac{k_- - (n-1)p}{\sqrt{(n-1)pq}} \approx (1-\epsilon)\sqrt{2}\log n$

$$= \frac{1}{\sqrt{8\pi p q n}} \int_{z=x\sqrt{2}}^{\infty} e^{-z^2/2}\, dz$$

$$\lesssim \frac{1}{\sqrt{8}\,\pi p q n} \cdot \frac{1}{x\sqrt{2}} \cdot n^{-2(1-\epsilon)^2}.$$

We get a similar bound for $\sum_{k_1} \Pr\left(\hat{d}(1) = k_1\right)^2$.

Thus

$$\Pr(\neg(\text{iii})) = O\left(n^{2-1-2(1-\epsilon)^2}\right)$$

$$= o(1).$$

$\square$

Edge Colouring

The Chromatic Index $\chi'(G)$ of graph G is the minimum number of colors that can be used to color the edges of G so that if 2 edges share a vertex, they have a different color.

Vizing's Theorem states that

$$\Delta(G) \leq \chi'(G) \leq \Delta(G)+1.$$

Also, if there is a unique vertex of maximum degree, then $\chi'(G) = \Delta(G)$.

So $\chi'(G_{n,p}) = \Delta(G_{n,p})$ whp.

Graph Isomorphism

In this section we describe a procedure for ordering the vertices of a graph G. If it succeeds then it is possible to quickly tell if $G \cong H$, for any H.

Algorithm

Input G. Parameter L.

Step 1

Re-label vertices so that degrees satisfy

$$d_G(v_1) \geq d_G(v_2) \geq \cdots \geq d_G(v_n)$$

If $\exists\, i \leq L$ such that $d_G(v_i) = d_G(v_{i+1})$: FAIL

Step 2

For $i > L$ let

$$X_i = \{ j \in \{1, 2, \ldots, L\} : (v_i, v_j) \in G \}$$

Re-label vertices so that these sets satisfy

$$X_{L+1} \succeq X_{L+2} \succeq \cdots \succeq X_n$$

— lexicographic ordering.

If $\exists\, i \geq L$ such that $X_i = X_{i+1}$: FAIL.

Suppose now that the above algorithm succeeds for G.

Given an n-vertex graph H we run the algorithm on H.

(i) If algorithm fails $G \not\cong H$.

(ii) Suppose ordering of $V(H)$ is $w_1, w_2, \ldots, w_n$. Then

$$G \cong H \iff v_i \to w_i \text{ is an isomorphism.}$$

Claim

Let $\rho = p^2 + q^2$ and $L = 3 \log_{1/\rho} n$.

Then whp the algorithm succeeds on $G = G_{n,p}$.

Proof

We have already proved that Step 1 succeeds whp.

We must now show that $X_i \neq X_j \ \forall i,j$ whp but there is slight problem because edges (v_i, v_j) are conditioned due to us knowing v_i has a high degree.

Fix i,j and let $\hat{G}_{i,j} = G \setminus \{i,j\}$.

Now if i,j are not high degree vertices then the L largest degree vertices in G, $\hat{G}_{i,j}$ will coincide, whp.

This is because there is whp, a gap of ≥ 10 between high vertex degrees in G.

Thus

$$\Pr(\text{Step 2 fails}) \leq$$

$$o(1) + \sum_{1 \leq i < j \leq n} \Pr\Big(i,j \text{ have same nbrs among } L \text{ largest degree vertices in } \hat{G}_{i,j} \Big\}$$

$$= o(1) + \binom{n}{2} \rho^{L}$$

$$= o(1).$$

$\square$

Automorphisms

It follows from the previous section that whp, $G_{n,p}$ has no non-trivial automorphisms.

For if $\sigma : [n] \to [n]$ is an automorphism, then

(i) $\sigma(v_i) = v_i$, $1 \leq i \leq L$
where v_i is the vertex with the i th largest degree.

(ii) $\sigma(v) = v$ for $v \notin \{v_1, v_2, \ldots, v_L\}$.

This is because all of the sets X_v are distinct.