Note Title 2/1/2006

Connectivity of random graphs

Let $p = \dfrac{\log n + c_n}{n}$. We prove

$$\lim_{n\to\infty} \Pr\left(G_{n,p} \text{ is connected}\right) = \begin{cases} 0 & c_n \to -\infty \\ e^{-e^{-c}} & c_n \to c \\ 1 & c_n \to +\infty \end{cases}$$

If $p_1 > p_2$ then we can write

$$G_{n,p_1} = G_{n,p_2} \cup G_{n,p_3}$$

where $$(1-p_1) = (1-p_2)(1-p_3)$$

and so

$$\Pr(G_{n,p_1} \text{ is connected})$$

$$\geq \Pr(G_{n,p_2} \text{ is connected})$$

can replace "is connected" by any monotone ↑ property.

It suffices to prove that

$$\Pr(G_{n,p} \text{ is connected}) \longrightarrow e^{-e^{-c}}$$

when $p = \frac{\log n + c}{n}$.

Now

$$\Pr(G_{n,p} \text{ is not connected})$$

$$= \Pr\left(\bigcup_{i=1}^{n/2} \exists \text{ a component of size } i \right)$$

So we have

$$\Pr(\exists \text{ isolated vertex}) \leq$$

$$\Pr(G_{n,p} \text{ is not connected}) \leq$$

$$\Pr(\exists \text{ isolated vertex}) + \sum_{k=2}^{n/2} \Pr\left(\exists \begin{array}{c}\text{component of} \\ \text{size } k\end{array}\right)$$

Now

$$\sum_{k=2}^{n/2} \Pr(\exists \text{ component of size } k)$$

$$\leq \sum_{k=2}^{n/2} E(\#\text{ of components of size } k)$$

$$\leq \sum_{k=2}^{n/2} \underbrace{\binom{n}{k} k^{k-2} p^{k-1} (1-p)^{k(n-k)}}_{u_k}$$

For $2 \leq k \leq 10$

$$u \leq e^k n^k \cdot \left(\frac{\log n + c}{n}\right)^{k-1} \cdot e^{-k(n-10)\frac{\log n + c}{n}} \leq (1+o(1)) \frac{e^{k(1-c)} (\log n)^{k-1}}{n^{k-1}}$$

and for $k \geq 10$

$$u_k \leq \left(\frac{ne}{k}\right)^k k^{k-2} \left(\frac{\log n + c}{n}\right)^{k-1} e^{-k(\log n + c)/2}$$

$$\leq n\left(\frac{e^{1-c/2+o(1)}\log n}{n^{1/2}}\right)^k$$

so

$$\sum_{k=2}^{n/2} u_k \leq (1+o(1))\frac{e^{-c}\log n}{n} + \sum_{k=10}^{n/2} n^{1+o(1)-k/2}$$

$$= O\left(n^{o(1)-1}\right).$$

It follows that

$$\Pr(G_{n,p} \text{ is connected}) = \Pr(\nexists \text{ an isolated vertex}) + o(1).$$

So now let

X_0 = the number of isolated vertices in $G_{n,p}$.

Then

$$E(X_0) = n(1-p)^{n-1}$$

$$= n \exp\{(n-1)\log(1-p)\}$$

$$= n \exp\left\{-(n-1)\sum_{k=1}^{\infty} \frac{p^k}{k}\right\}$$

$$= n \exp\left\{-(\log n + c) + O\left(\frac{(\log n)^3}{n}\right)\right\}$$

$$\approx e^{-c}.$$

If we let

A_i be the event $\{$vertex i is isolated$\}$

and if $$S_t = \sum_{\substack{X \subseteq [n] \\ |X| = t}} \Pr(A_X)$$

then $$S_t = \binom{n}{t} (1-p)^{t(n-t) + \binom{t}{2}}$$

$$\approx e^{-tc}/t! \qquad t = O(1).$$

Thus we deduce, as in our study of isolated lines,

that $$\lim_{n\to\infty} \Pr(X_0 = 0) = e^{-e^{-c}}.$$

Hitting Time Version in Graph Process

Let
$$m_1^* = \min\{m : \delta(G_m) \geq 1\}$$
$$m_c^* = \min\{m : G_m \text{ is connected}\}$$

We show

$$m_1^* = m_c^* \quad \text{whp}$$

Let

$$m_{\pm} = \frac{1}{2} n \log n \pm \frac{1}{2} n \log\log n$$

and

$$p_{\pm} = \frac{m}{N} \approx \frac{\log n \pm \log\log n}{n}$$

We first show that whp

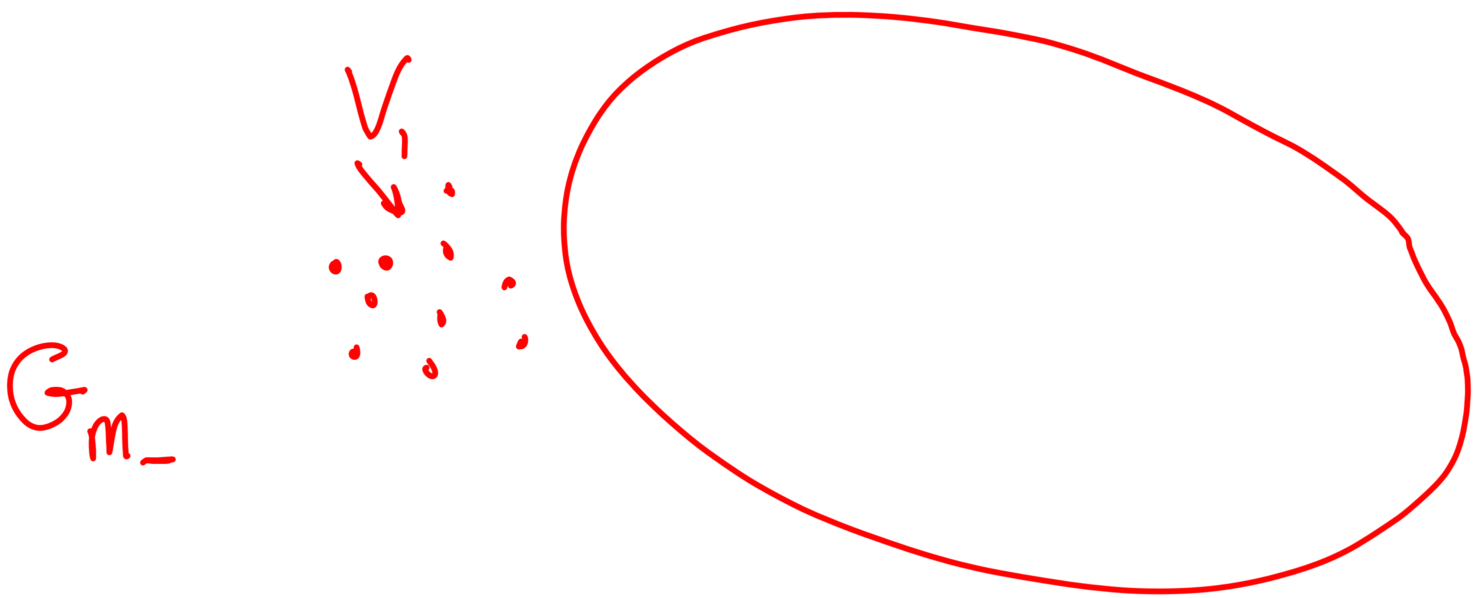
(i) G_{m_-} consists of a giant connected component plus a set V_1 of $\leq 2\log n$ vertices.

(ii) G_{m_+} is connected.

Assume (i) and (ii).

It follows that whp

$$m_- \le m_1^* \le m_c^* \le m_+$$



To create G_{m_+} we add $m_+ - m_-$ random edges.

$m_1^* = m_c^*$ if none of these edges is contained in V_1

Thus

$$\Pr\left(m_1^* < m_c\right) \leq O(1) + (m_+ - m_1)\,\frac{\frac{1}{2}|V_1|^2}{N - m_+}$$

$$= O(1) + \frac{n\,(\log\log n) * \left(2\,(\log n)^2\right)}{\frac{1}{2}n^2 - O(n\log n)}$$

$$= O(1).$$

(1) Let $p_- = \dfrac{m_-}{N} \approx \dfrac{\log n - \log\log n}{n}$

and let X_1 = # isolated vertices in G_{n,p_-}.

Then

$$E(X_1) = n(1-p_-)^{n-1}$$

$$= n e^{-np + O(np^2)}$$

$$\approx \log n.$$

$$E(X_1^2) = E(X_1) + n(n-1)(1-p)^{2n-3}$$

$$\leq E(X_1) + E(X_1)^2 (1-p)^{-3}$$

so

$$\mathrm{Var}(X_1) \leq E(X_1) + 4E(X)^2 p$$

$$\Pr(X_1 \geq 2\log n) = \Pr(|X_1 - E(X_1)| \geq (1+o(1))E(X_1))$$

$$\leq (1+o(1))\left(\frac{1}{E(X_1)} + 4p\right)$$

$$= o(1).$$

Having $\geq 2\log n$ isolated vertices is a monotone property and so whp G_{m_-} has $< 2\log n$ isolated vertices.

To show that the rest of G_{m_-} is a single component we let X_k, $2 \leq k \leq \frac{n}{2}$ be the number of components with k vertices in G_{p_-}.

Repeating the calculation on p5

$$E\left(\sum_{k=2}^{n/2} X_k\right) = O(n^{o(1)-1})$$

Let $\mathcal{E} = \{\exists$ component of size $2 \le k \le \frac{1}{2}n\}$

$$\Pr(G_{m_-} \in \mathcal{E}) \le O(\sqrt{n}) \Pr(G_{n,p_-} \in \mathcal{E})$$

$$= o(1)$$

and this complete proof of (i).

(ii) G_{m_+} is connected whp.

This follows from $G_{n,p}$ is connected whp for $np - \log n \to \infty$

or by implication G_m is connected whp if

$$n \cdot \frac{m}{N} - \log n \to \infty$$

$$\frac{n\, m_+}{N} = \frac{n\left(\frac{1}{2} n \log n + \frac{1}{2} n \log\log n\right)}{N}$$

$$\approx \log n + \log\log n.$$

k - connectivity.

Here we will prove that if $k = O(1)$ and

$$m = \frac{1}{2} n \left(\log n + (k-1) \log\log n + c_n \right)$$

then

$$\lim_{n \to \infty} \Pr(G_{n,m} \text{ is } k\text{-connected}) = \begin{cases} 0 & c_n \to -\infty \\ e^{-\frac{e^{-c}}{(k-1)!}} & c_n \to c \\ 1 & c_n \to \infty \end{cases}$$

Let $p = \dfrac{\log n + (k-1)\log\log n + c}{n}$

We will prove

(i) $E(\#\text{ vertices of degree} \le k-2) = o(1)$

(ii) $E(\#\text{ vertices of degree } k-1) \approx \dfrac{e^{-c}}{(k-1)!}$

It then a simple matter to verify that

$$\Pr(\delta(G_{n,p}) \geq k) \approx e^{-\frac{e^{-c}}{(k-1)!}}$$

$E(\#\text{ vertices of degree } t \leq k-1)$

$$= n \binom{n-1}{t} p^t (1-p)^{n-1-t}$$

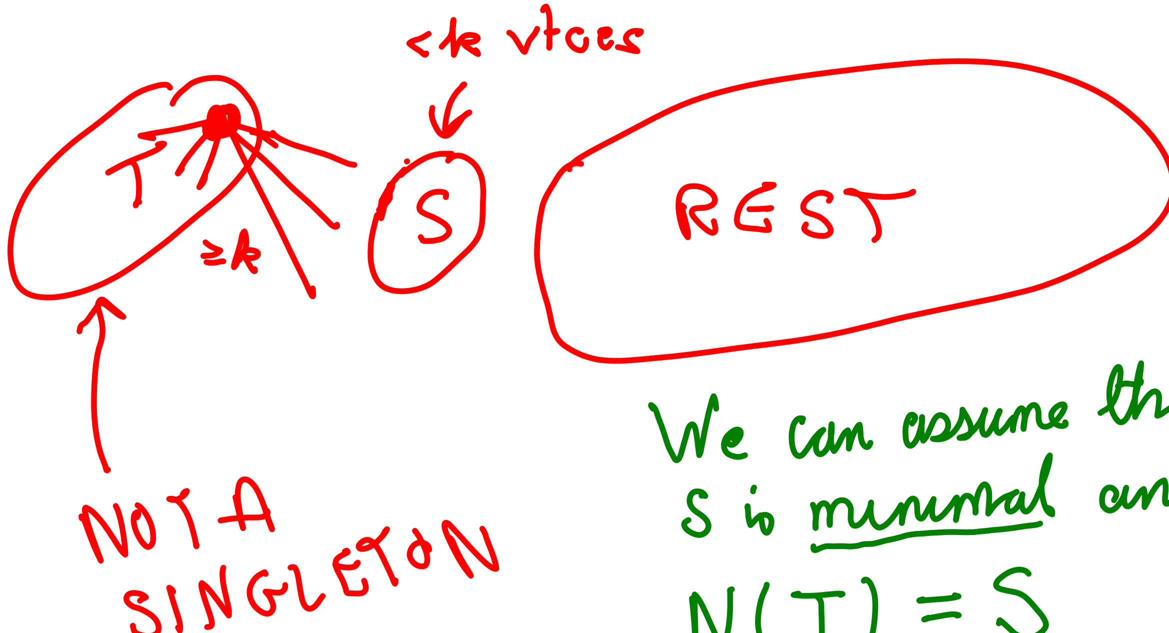
$$\simeq n \cdot \frac{n^t}{t!} \cdot \frac{(\log n)^t}{n^t} \cdot \frac{e^{-c}}{n (\log n)^{k-1}}$$

and (i) and (ii) follow immediately.

We now show that,

$$\Pr\big(\exists\, S,\ |S| < k \text{ and } T,\ k-|S|+1 \le |T| \le \tfrac{1}{2}(n-s)$$
$$T \text{ is a component of } G_{n,p} \setminus S\big) = o(1).$$

This implies that if $\delta(G_{n,p}) \ge k$ then it is k-connected whp



We can assume that S is <u>minimal</u> and then

$N(T) = S$

First moment :

$E(\# S, T) \leqslant$

Case 1 : $s+2 \leqslant t \leqslant \log n$

$$\sum_{s=0}^{k-1} \sum_{t=s+2}^{\log n} \binom{n}{s}\binom{n}{t} t^{t-2} p^{t-1} (1-p)^{t(n-s-t)}$$

$$\leqslant \sum_{s=0}^{k-1} \sum_{t=s+2}^{\log n} n^s \cdot \left(\frac{ne}{t}\right)^t \cdot t^{t-2} \cdot \left(\frac{e^{o(1)} \log n}{n}\right)^{t-1} \frac{e^{O((\log n)^2/n)}}{n^t (\log n)^{(k-1)t}}$$

$$\leqslant \sum_{s=0}^{k-1} \sum_{t=s+2}^{\log n} \left(e^{1+o(1)} \log n\right)^t n^{s-t}$$

$= O(1)$.

Case 2 : $t > \log n$

$$\sum_{s=0}^{k-1} \sum_{t=\log n}^{\frac{1}{2}(n-s)} \binom{n}{s}\binom{n}{t} t^{t-2} p^{t-1} (1-p)^{t(n-s-t)}$$

$$\leq \sum_{s=0}^{k-1} \sum_{t=\log n}^{\frac{1}{2}(n-s)} n^s \left(\frac{ne}{t}\right)^t t^{t-2} \left(\frac{e^{o(1)} \log n}{n}\right)^{t-1} n^{-t/2}$$

$$\leq \sum_{s=0}^{k-1} \sum_{t=\log n}^{\frac{1}{2}(n-s)} n^{1+s-\frac{1}{2}t} \left(e^{1+o(1)} \log n\right)^t$$

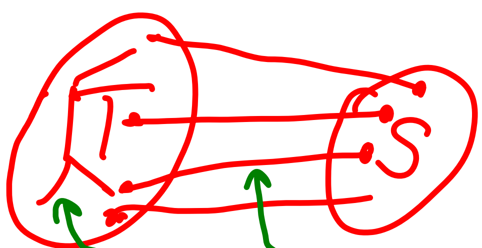
$$= o(1).$$

Case 3 : $k-s+1 \le t \le s+1$

$$\sum_{s=0}^{k-1} \sum_{\substack{t \ge 2 \\ t \ge k-s+1}}^{s+1} \binom{n}{s}\binom{n}{t} t^{t-2} \binom{st}{s} p^{t-1+s} (1-p)^{t(n-s-t)}$$

$$\le \sum_{s=0}^{k-1} \sum_{t} n^{s+t} 2^{st} \left(\frac{e^{O(1)} \log n}{n}\right)^{t-1+s} \frac{1+o(1)}{n^t}$$

$= o(1).$



edges $\ge t-1+s$

$\square$