Structure of graph when
\[ m = \frac{1}{2} cn, \quad c > 1 \text{ constant.} \]

We will work in \( G_{n, p} \)
\[ p = \frac{c}{n} = \frac{m}{N} \]
Suppose now that $X_k$ is the number of components of size $k$. Then

$$
\mathbb{E}(X_k) \leq \binom{n}{k} k^{k-2} p^{k-1} (1-p)^{k(n-k)}
$$

\[
\leq \frac{A}{\sqrt{k}} \left( \frac{ne}{k} \right)^k e^{-k^2/2n} k^{k-2} \left( \frac{c}{n} \right)^{k-1} e^{-ck + ck^2/n}
\]

\[
\leq \frac{An}{k^{5/2}} \left( ce^{1-c + ck/n} \right)^k.
\]
Now let $B_1 = B_1(c)$ be small enough so that \( c e^{1-c + B_1} < 1 \).

and let $B_0 = B_0(c)$ be large enough so that \( (c e^{1-c + o(1)}) B_0 \log n < \frac{1}{n^2} \).

It follows that with a component of size $k \in [B_0 \log n, B_1 n]$
Our calculations for $c < 1$ can be repeated to show that if

$$\alpha = c - 1 - \log c$$

**Theorem**

Suppose $w \to \infty$

(i) Whp $\exists$ an isolated line $g$, size

$$\frac{1}{\alpha} \left( \log n - \frac{5}{2} \log \log n \right) - w \leftarrow k^-.$$

(ii) Whp $\exists$ an isolated line $g$, size

$$\geq \frac{1}{\alpha} \left( \log n - \frac{5}{2} \log \log n \right) + w \leftarrow k^+$$

provided $w = O(\log n)$. 
We can say a little more about components of size \( k \), \( k = O(\log n) \).

If we repeat the calculations for \( c < 1 \) then we find that if \( Y_k \)

is the number of isolated lines of size \( k \)

\[ k = \frac{1}{\alpha} \left( \log n - \frac{5}{2} \log \log n \right) - w \]

then

\[ E(Y_k) \geq A e^{\alpha w} \]

for some \( A = A(c) > 0 \).
\[ E(Y_k^2) \leq E(Y_k) + E(Y_k)^2 (1-p)^{-k} \leq \]

\[ \text{Var}(Y_k) \leq E(Y_k) + E(Y_k)^2 (1-p)^{-k} - 1 \leq \]

\[ E(Y_k) + 2E(Y_k)^2 c k / n. \]

So,

\[ P\left( |Y_k - E(Y_k)| \geq \epsilon E(Y_k) \right) \leq \frac{1}{e^{2E(Y_k)}} + \frac{2c k}{\epsilon^2 n}. \quad (\star) \]
We now estimate the total number of vertices on small tree components i.e. size $\leq B_0 \log n$.

(i) $1 \leq k \leq k_0 = \frac{1}{2\alpha} \log n$

$$E\left(\sum_{k=1}^{k_0} k Y_k \right) = \frac{n}{c} \sum_{n=1}^{k_0} \frac{k^{k-1}}{k!} (ce^{-c})^k$$

$$\sum_{n=1}^{\infty} \frac{n}{c} \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} (ce^{-c})^k$$

Since $\frac{k^{k-1}}{k!} < e^k$ and $ce^{1-c} < 1$. 
Putting $\varepsilon = \frac{1}{\log n}$ we see that the probability that any $\frac{k}{k}$ deviates from its mean by more than $1 \pm \varepsilon$ is at most $(\text{see (4) on p} \ 6) \left[ \sum_{k=1}^{k_0} \left[ \frac{(\log n)^2}{n^{1/3}} + O\left( \frac{(\log n)^3}{n} \right) \right] = o(1) \right]$. 
Thus we have

\[
\sum_{k=1}^{k_0} k Y_k = \frac{n}{c} \sum_{k=1}^{k_0} \frac{k^{k-1}}{k!} (ce^{-c})^k
\]

\[
\leq \frac{n}{c} \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} (ce^{-c})^k
\]

\[
= \frac{n}{c} \sum_{k=1}^{\infty} \frac{(xe-x)^k}{k!}
\]

where \(0 < x < 1\) and \(xe^{-x} = ce^{-c}\)

\[
= \frac{nx}{c}.
\]
Now consider $k_o < k \leq B_o \log n$.

\[ E \left( \sum_{k=k_o+1}^{B_o \log n} k Y_k \right) \leq \]

\[ \frac{n}{c} \sum_{k=k_o+1}^{B_o \log n} \frac{A n}{k^{3/2}} \left( ce^{1-c + ck \log n} \right)^k \]

\[ = o \left( \frac{n}{(\log n)^{3/2}} \right). \]

So, by the Markov inequality, \( Y_k \), \( k = k_o + 1 \)

\[ \sum_{k=k_o+1}^{B_o \log n} k Y_k = o(n). \]
Now consider the number of vertices $Z_k$ on non-tree components with $k$ vertices, $1 \leq k \leq B_0 \log n$.

$$E\left(\sum_{k=1}^{B_0 \log n} Z_k\right) \leq \sum_{k=1}^{B_0 \log n} \binom{n}{k} k^{k-2} \left(\frac{k}{n}\right)^k \left(1 - \frac{c}{n}\right)^{k(n-k)}$$

$$\leq \sum_{k=1}^{B_0 \log n} \left(ce^{1-c+k/n}\right)^k$$

$$= O(1).$$

So, by the Markov inequality, whp

$$\sum_{k=1}^{B_0 \log n} Z_k = o(n).$$
So far: Why
there are \( \lambda x e^{-\frac{1}{c}} \) vertices on components of size \( k \),
\( 1 \leq k \leq \beta_0 \log n \).
The giant component.

Let $c_1 = c - \frac{\log n}{n^2}$ and $p_1 = \frac{c_1}{n}$ and define $p_2$ by

$$1 - p = (1 - p_1)(1 - p_2).$$

Then

$$G_{n,p} = G_{n,p_1} \cup G_{n,p_2}$$

since probability $e$ is not included in

$$(1 - p_1)(1 - p_2).$$

Note that

$$p_2 \geq \frac{\log n}{n^2}.$$
If \( x_1 e^{-x_1} = c_1 e^{-c_1} \) then \( x_1 \leq c_1 \) and so by our previous analysis, why, \( G_{n,p_1} \) has no components of size in the range \([B_0 \log n, B_0 n]\).

Suppose there are components \( C_1, C_2, \ldots, C_l \) with \( |C_i| > B_0 n \). Thus \( l \leq \frac{1}{B_0} \).

Now we add in the edges of \( G_{n,p_2} \).
\[
\Pr( \exists i, j : \text{no } G_{n, p_2} \text{ edge joins } C_i, C_j ) \\
\leq \left( \frac{\ell}{2} \right) (1 - p_2)^{2 (3^n)} \\
\leq \ell^2 e^{- \theta_0^2 (\log n)^2} \\
= o(1).
\]

So why \( G_{n, p} \) has a unique component of size greater than \( \theta_0 \log n \), and it is of size \( (1 - \frac{\theta}{e}) n \).
Duality

Let \( N = \frac{n \lambda}{\alpha} \approx \# \text{vertices outside giant component}. \)

Let \( q = \frac{n}{N} (= p). \)

Note that \( x < 1 \) and \# isolated trees of size \( k \) is

\[
\leq \frac{n}{c} \cdot \frac{\lambda^{k-2}}{k!} \cdot (c e^{-c})^k \\
= \frac{n}{c} \cdot \frac{\lambda^{k-2}}{k!} \cdot (c e^{-c})^k \\
= \frac{N}{c} \cdot \frac{\lambda^{k-2}}{k!} \cdot (c e^{-c})^k.
\]

Thus graph outside of giant component is asymptotically equal to \( G_{N, \lambda/N} \) in distribution.