Note Title 1/27/2006

Structure of graph when

$m = \frac{1}{2}cn$, $c > 1$ constant.

We will work in $G_{n,p}$

$$p = \frac{c}{n} \approx \frac{m}{N}$$

Suppose now that X_k is the number of components of size k. Then

$$E(X_k) \leq \binom{n}{k} k^{k-2} p^{k-1} (1-p)^{k(n-k)}$$

$$\leq \frac{A}{\sqrt{k}} \left(\frac{ne}{k}\right)^k e^{-k^2/2n} k^{k-2} \left(\frac{c}{n}\right)^{k-1} e^{-ck+ck^2/n}$$

$$\leq \frac{An}{k^{5/2}} \left(ce^{1-c+ck/n}\right)^k.$$

Now let $B_1 = B_1(c)$ be small enough so that

$$ce^{1-c+B_1} < 1.$$

and let $B_0 = B_0(c)$ be large enough so that

$$\left(ce^{1-c+o(1)}\right)^{B_0 \log n} < \frac{1}{n^2}.$$

It follows that whp $\nexists$ a component of size

$$k \in [B_0 \log n, B_1 n]$$

Our calculations for $c<1$ can be repeated to show that if

$$\alpha = c-1-\log c$$

<u>Theorem</u>

Suppose $\omega \to \infty$

(i) Whp $\exists$ an isolated tree of size

$$\frac{1}{\alpha}\left(\log n - \frac{5}{2}\log\log n\right) - \omega \leftarrow k_-$$

(ii) Whp $\nexists$ an isolated tree of size

$$\geq \frac{1}{\alpha}\left(\log n - \frac{5}{2}\log\log n\right) + \omega \leftarrow k_+$$

provided $\omega = O(\log n)$.

We can say a litlile more about components of size k, $k = O(\log n)$.

If we repeat the calculations for $c<1$ then we find that if Y_k is the number of isolated trees of size

$$k = \frac{1}{\alpha}\left(\log n - \frac{5}{2}\log\log n\right) - \omega$$

then

$$E(Y_k) \geq A e^{\alpha \omega}$$

for some $A = A(c) > 0$.

$$E(Y_k^2) \leq E(Y_k) + E(Y_k)^2 (1-p)^{-k}$$

So

$$\mathrm{Var}(Y_k) \leq E(Y_k) + E(Y_k)^2 \left((1-p)^{-k} - 1\right)$$

$$\leq E(Y_k) + 2E(Y_k)^2 ck/n.$$

So

$$\Pr\left(\left|Y_k - E(Y_k)\right| \geq \epsilon E(Y_k)\right)$$

$$\leq \frac{1}{\epsilon^2 E(Y_k)} + \frac{2ck}{\epsilon^2 n}. \quad (*)$$

We now estimate the total number of vertices on small tree components i.e. size $\leq B_0 \log n$.

(i) $1 \leq k \leq k_0 = \frac{1}{2\alpha} \log n$

$$E\left(\sum_{k=1}^{k_0} k Y_k\right) \approx \frac{n}{c} \sum_{k=1}^{k_0} \frac{k^{k-1}}{k!} (ce^{-c})^k$$

$$\approx \frac{n}{c} \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} (ce^{-c})^k$$

since $\frac{k^{k-1}}{k!} < e^k$ and $ce^{1-c} < 1$.

Putting $\epsilon = \dfrac{1}{\log n}$ we see that the probability that any Y_k deviates from its mean by more than $1 \pm \epsilon$ is at most (see (*) on p 6)

$$\sum_{k=1}^{k_0} \left[\frac{(\log n)^2}{n^{1/3}} + O\left(\frac{(\log n)^3}{n} \right) \right] = o(1).$$

Thus whp

$$\sum_{k=1}^{k_0} k Y_k \approx \frac{n}{c} \sum_{k=1}^{k_0} \frac{k^{k-1}}{k!} \left(ce^{-c}\right)^k$$

$$\approx \frac{n}{c} \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} \left(ce^{-c}\right)^k$$

$$= \frac{n}{c} \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} \left(xe^{-x}\right)^k$$

where $0 < x < 1$ and $xe^{-x} = ce^{-c}$

$$= \frac{nx}{c}.$$

Now consider $k_0 < k \leq B_0 \log n$.

$$E\left(\sum_{k=k_0+1}^{B_0 \log n} k Y_k\right) \leq$$

$$\frac{n}{c} \sum_{k=k_0+1}^{B_0 \log n} \frac{An}{k^{3/2}} \left(ce^{1-c} + ck/n\right)^k$$

$$= o\left(n / (\log n)^{3/2}\right).$$

So, by the Markov inequality, whp,

$$\sum_{k=k_0+1}^{B_0 \log n} k Y_k = o(n).$$

Now consider the number of vertices Z_k on non-tree components with k vertices, $1 \le k \le B_0 \log n$.

$$E\left(\sum_{k=1}^{B_0 \log n} Z_k\right) \le \sum_{k=1}^{B_0 \log n} \binom{n}{k} k^{k-2} \binom{k}{2} \left(\frac{c}{n}\right)^k \left(1-\frac{c}{n}\right)^{k(n-k)}$$

$$\le \sum_{k=1}^{B_0 \log n} \left(c e^{1-c+k/n}\right)^k$$

$$= O(1).$$

So, by the Markov inequality, whp

$$\sum_{k=1}^{B_0 \log n} Z_k = o(n).$$

So far: whp

there are $\approx \frac{nx}{c}$ $\qquad xe^{-x} = ce^{-c}$

vertices on components of size k,

$1 \leq k \leq \beta_0 \log n$.

The giant component.

Let $c_1 = c - \frac{\log n}{n^2}$ and $p_1 = \frac{c_1}{n}$

and define p_2 by

$$1 - p = (1 - p_1)(1 - p_2).$$

Then

$$G_{n,p} = G_{n,p_1} \cup G_{n,p_2}$$

since probability e is not included in $G_{n,p_1} \cup G_{n,p_2}$ is $(1-p_1)(1-p_2)$.

Note that

$$p_2 \geqslant \frac{\log n}{n^2}.$$

If $x_1 e^{-x_1} = c_1 e^{-c_1}$ then $x_1 \simeq x$ and so by our previous analysis, whp, G_{n,p_1} has no components of size in the range $[B_0 \log n, B_1 n]$.

Suppose there are components $C_1, C_2, \dots, C_\ell$ with $|C_i| > B_0 n$. Thus $\ell \leq \frac{1}{B_0}$.

Now we add in the edges of G_{n,p_2}.

$$\Pr(\exists\, i,j : \text{no } G_{n,p_2} \text{ edge joins } C_i, C_j)$$

$$\leq \binom{l}{2} (1-p_2)^{(B_0 n)^2}$$

$$\leq l^2 e^{-B_0^2 (\log n)^2}$$

$$= o(1).$$

So whp $G_{n,p}$ has a unique component of size greater than $B_0 \log n$, and it is of size $\left(1 - \frac{x}{c}\right) n$.

Duality

Let $N = \frac{nx}{c} \approx$ # vertices outside giant whp.

Let $q = \frac{x}{N}$ $(\leftarrow p)$.

Note that $x < 1$ and # of isolated trees of size k is whp

$$\approx \frac{n}{c} \cdot \frac{k^{k-2}}{k!} (ce^{-c})^k$$

$$= \frac{N}{x} \cdot \frac{k^{k-2}}{k!} (xe^{-x})^k.$$

Thus graph outside of giant component is asymptotically equal to $G_{N, \frac{x}{N}}$ in distribution.