Structure of graph when

$m = \frac{1}{2} cn$, $0 < c < 1$ constant.

We will work in $G_{n,p}$

$$p = \frac{c}{n} \approx \frac{m}{N}$$

Note Title

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Cycles

Whp the are $\leq \log n$ edges on cycles.

Let $X_k = \#$ cycles of length k.

$$E(X_k) = \binom{n}{k} \frac{(k-1)!}{2} p^k$$

$$< \frac{n^k}{k!} \frac{(k-1)!}{2} \left(\frac{c}{n}\right)^k$$

$$= \frac{c^k}{2k}.$$

So if $X = 3X_3 + 4X_4 + \cdots + nX_n$

$\Rightarrow$ # edges on cycles

then

$$E(X) \leq \sum_{k=3}^{n} k \cdot \frac{c^k}{2k} < \frac{1}{1-c}.$$

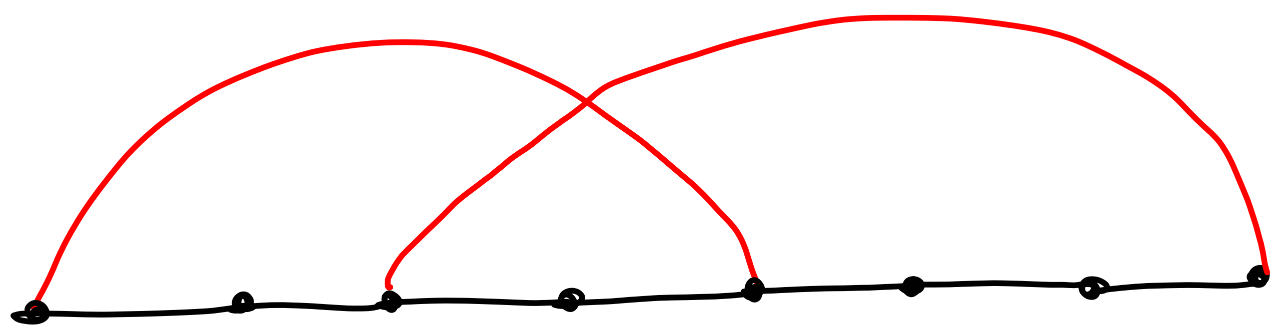
Applying the Markov inequality gives

$$\Pr(X > \log n) \leq \frac{1}{(\log n)(1-c)} = o(1).$$

Claim: whp $\nexists$ a pair of cycles that are in the same component

Proof

If a pair exists then there is a <u>minimal</u> pair C_1, C_2



$$E(\# C_1, C_2) \leq \sum_{k \geq 3} \binom{n}{k} \cdot \frac{k!}{2} k^2 p^{k+1}$$

$$\leq \frac{1}{n} \sum_{k \geq 3} c^{k+1} k^2$$

$$\to 0.$$

So whp every component contains at most one cycle.

We now show that whp size of largest component is $O(\log n)$.

Let X_k be the number of components of size k that are unicyclic

$$E(X_k)$$

$$\leq \binom{n}{k} k^{k-2} \binom{k}{2} p^k (1-p)^{k(n-k)+\binom{k}{2}-k}$$

$$\leq \frac{n^k}{k!} e^{-\frac{k(k-1)}{2n}} k^k \frac{c^k}{n^k} e^{-ck+ck(k-1)/2n+ck/2n}$$

$$\binom{n}{k} = \frac{n^k}{k!} \prod_{i=0}^{k-1} \left(1-\frac{i}{n}\right) \quad \& \quad 1-x \leq e^{-x}$$

$$\leq \frac{n^k}{k!} e^{-\frac{k(k-1)}{2n}} k^k \frac{c^k}{n^k} e^{-ck + \frac{ck(k-1)}{2n} + c/2}$$

$$\leq (ce^{1-c})^k e^{c/2}.$$

So if $\omega \to \infty$,

$\Pr(\exists$ unicyclic component of size $\geq \omega)$

$$\leq \sum_{k=\omega}^{n} e^{c/2} (ce^{1-c})^k$$

$\to 0$

since $ce^{1-c} < 1$ for $c \neq 1$.

Now let X_k be the number of isolated trees.

Let
$$\alpha = c - 1 - \log c$$

Theorem

Suppose $\omega \to \infty$

(i) Whp $\exists$ an isolated tree of size

$$\frac{1}{\alpha}\left(\log n - \frac{5}{2}\log\log n\right) - \omega \quad \leftarrow k_-$$

(ii) Whp $\nexists$ an isolated tree of size

$$\geq \frac{1}{\alpha}\left(\log n - \frac{5}{2}\log\log n\right) + \omega \quad \leftarrow k_+$$

Now let X_k = number of isolated trees of size k.

$$E(X_k) = \binom{n}{k} k^{k-2} p^{k-1} (1-p)^{k(n-k)+\binom{k}{2}-k+1}.$$

(i) Suppose $k = O(\log n)$. Then

$$E(X_k) \approx \frac{(1+o(1))}{\sqrt{2\pi k}} \left(\frac{ne}{k}\right)^k k^{k-2} \left(\frac{c}{n}\right)^{k-1} e^{-ck}$$

$$= \frac{(1+o(1))}{\sqrt{2\pi}} \frac{n}{k^{5/2}} \left(ce^{1-c}\right)^k.$$

Putting $k = k_-$

we see that

$$E(X_k) = \frac{(1+o(1))}{\sqrt{2\pi}} \frac{n}{k^{5/2}} \left(ce^{1-c}\right)^k.$$

$$= \frac{(1+o(1))}{\sqrt{2\pi}} \cdot \frac{n}{k^{5/2}} \cdot \frac{(\log n)^{5/2} e^{\alpha w}}{n}$$

$$\geq A e^{\alpha w}.$$

We continue via second moment method.

$$E(X_k^2) \le E(X_k)\left(1 + (1-p)^{-k} E(X_k)\right)$$

[Same argument as for fixed tree T of size k]

Thus

$$\frac{E(X_k)^2}{E(X_k^2)} \ge 1 - \frac{1}{2Ae^{\alpha w}} \longrightarrow 1.$$

and we have (i).

For (ii) we go back to

$$E(X_k) = \binom{n}{k} k^{k-2} p^{k-1} (1-p)^{k(n-k)+\binom{k}{2}-k+1}$$

$$\leq \frac{A}{\sqrt{k}} \left(\frac{ne}{k}\right)^k e^{-k^2/2n} k^{k-2} \left(\frac{c}{n}\right)^{k-1} e^{-ck+ck^2/2n}$$

$$\leq \frac{An}{k^{5/2}} (ce^{1-c})^k$$

and then

$$\sum_{k=k_+}^{n} E(X_k) \leq An \sum_{k=k_+}^{n} \frac{(ce^{1-c})^k}{k^{5/2}} = o(1).$$

Useful Identity

$0 \leq c \leq 1$ implies $\frac{1}{c} \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} (ce^{-c})^k = 1.$

Proof

Assume first $c < 1$.

Let σ = number of vertices of $G_{n,p}$ that lie on unicyclic components.

$$n = \sum_{k=1}^{n} k X_k + \sigma$$

# of trees of size k.

so

$$n = \sum_{k=1}^{n} k E(X_k) + E(\sigma)$$

(i) $E(\sigma) \leq \log n$

(ii) $$\sum_{k \geq k_+} k E(X_k) \leq \frac{1}{c} \sum_{k=k_+}^{n} \frac{(ce^{1-c})^k}{k^{3/2}} = O(1).$$

(iii) If $k < k_+$ then

$$E(X_k) = \binom{n}{k} k^{k-2} p^{k-1} (1-p)^{k(n-k)+\binom{k}{2}-k+1}.$$

$$= (1+o(1)) \frac{n}{c} \frac{k^{k-1}}{k!} (ce^{-c})^k$$

So

$$n = \sum_{k=1}^{n} k E(X_k) + E(\sigma)$$

$$= O(n) + \frac{n}{c} \sum_{k=1}^{k^+} \frac{k^{k-1}}{k!} (c e^{-c})^k$$

$$= O(n) + \frac{n}{c} \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} (c e^{-c})^k$$

Now divide through by n.

$\square$