Shortest Paths

Let the arcs of the complete digraph $D_n$ on $[n]$ be given independent lengths $X_e$, $e \in [n]^2$.

Here $X_e$ is exponential with mean 1 i.e., $P(X_e \geq t) = e^{-t}$ for all $t \geq 0$. 
Theorem

Let \( X_{ij} = \text{distance from } i \text{ to } j \). Then

(i) For any fixed \( i, j \),
\[
P\left( \left| \frac{X_{ij}}{\log n/n} - 1 \right| \geq \varepsilon \right) \rightarrow 0, \quad \forall \varepsilon > 0
\]

(ii) For any fixed \( i \),
\[
P\left( \left| \frac{Z_i}{2 \log n/n} - 1 \right| \geq \varepsilon \right) \rightarrow 0, \quad \forall \varepsilon > 0
\]

Here \( Z_i = \max_j X_{ij} \).
Proof

Two main properties of exponential $X$:

(P1) $\Pr(X > \alpha + \beta | X > \alpha) = \Pr(X > \beta)$.

(P2) If $X_1, X_2, \ldots, X_m$ are independent exponentials then $\min \{X_1, X_2, \ldots, X_m\}$ is an exponential with mean $\frac{1}{m}$.
Fix \( i = 1 \) and consider Dijkstra's shortest path algorithm. This produces a tree.

\[ v_1, v_2, v_3, v_4, v_5 \]

Suppose that vertices are added to the tree in the order \( v_1, v_2, ..., v_n \) and that \( \text{dist}(v_1, v_i) = Y_j \).
It follows from P1 (p3) that

\[ Y_{k+1} = \min_{i=1, \ldots, k} \left[ Y_i + X(v_i, v) \right] \leq Y_k \]

\[ = Y_k + \text{Exponential} \]

So \( Y_{k+1} = Y_k + E_k \)

where \( E_k \) is exponential with mean \( \frac{1}{k(n-k)} \) and is independent of \( Y_k \).
So

\[ E(Y_n) = \sum_{k=1}^{n-1} \frac{1}{k(n-k)} \]

\[ = \frac{1}{n} \sum_{k=1}^{n-1} \left( \frac{1}{k} + \frac{1}{n-k} \right) \]

\[ = \frac{2}{n} \sum_{k=1}^{n-1} \frac{1}{k} \]

\[ \sim \frac{2 \log n}{n} \]

Also

\[ \text{Var}(Y_n) = \sum_{k=1}^{n-1} \text{Var}(E_{kn}) = \sum_{k=1}^{n-1} \left( \frac{1}{k(n-k)} \right)^2 \]

\[ \leq 2 \sum_{k=1}^{n/2} \left( \frac{1}{k(n-k)} \right)^2 \leq \frac{8}{n^2} \sum_{k=1}^{n/2} \frac{1}{k^3} = O(n^{-2}) \]

and we can use Chebyshev to prove (ii).
Now fix $j = 2$. Then $y_i$ is defined by $\mathcal{N}_j = 2$, we see that $y_i$ is uniform over $\{2, 3, \ldots, n\}$.

So,

$$E(X_{1,2}) = \frac{1}{n-1} \sum_{i=2}^{n} \sum_{k=1}^{i} \frac{1}{k \ln(n-k)}$$

$$= \frac{1}{n-1} \sum_{k=1}^{n-1} \frac{n-k}{k \ln(n-k)}$$

$$= \frac{\log_{e/n} n}{n}.$$
For variance we have

\[ X_{1,2} = \delta_2 Y_2 + \delta_3 Y_3 + \ldots + \delta_n Y_n \]

where

\[ \delta_i \in \{0, 1\} ; \quad \delta_2 + \ldots + \delta_n = 1 ; \quad P(\delta_i = 1) = \frac{1}{n-1} . \]

\[
\text{Var} (X_{1,2}) = \sum_{i=2}^{n} \text{Var} (\delta_i Y_i) \\
+ \sum_{i \neq j} \text{Cov} (\delta_i Y_i; \delta_j Y_j) \\
\leq \sum_{i=2}^{n} \text{Var} (\delta_i Y_i) \\
\text{Cov} (\delta_i Y_i; \delta_j Y_j) = E(\delta_i Y_i; \delta_j Y_j) - E(\delta_i Y_i) E(\delta_j Y_j) \leq 0
\]
\[ \text{Var} (X_{1,2}) \leq \sum_{i=2}^{n} \text{Var} (S_i, Y_i) \]
\[ \leq \sum_{i=2}^{n} \frac{1}{n-1} \sum_{k=1}^{i-1} \left( \frac{1}{k} \frac{1}{(n-k)} \right)^2 \]
\[ = \mathcal{O} \left( \frac{1}{n^2} \right). \]

We can now use Chebyshev.