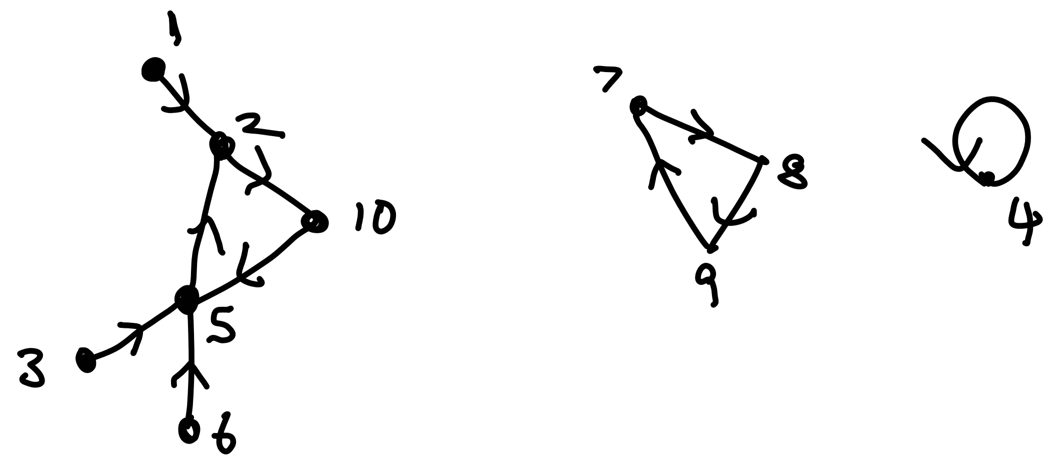
Random Mappings

Let f be chosen uniformly at random from the set of all n^n mappings from $[n] \to [n]$.

Let D_f be the digraph $([n], (x, f(x)))$ and let G_f be obtained from D_f by ignoring orientation.

Ex:

x	1	2	3	4	5	6	7	8	9	10
$f(x)$	2	10	5	4	2	5	8	9	7	5



In general D_f consists of unicyclic components, where each such consists of a directed cycle C with trees rooted at each vertex of C.

Thm 1

$$\Pr(G_f \text{ is connected}) \approx \sqrt{\frac{\pi}{2n}}$$

Proof

Let $T(n,k)$ denote the number of forests with vertex set $[n]$, k trees, in which $1, 2, \ldots, k$ are in different trees. We show later that

$$T(n,k) = k n^{n-k-1}.$$

$$\Pr(G_f \text{ is connected}) =$$

$$n^{-n}\sum_{k=1}^{n} \underbrace{\binom{n}{k}(k-1)!}_{\text{choose cycle of length } k} \; \underbrace{T(n,k)}_{\text{finish off mapping}}$$

$$= \frac{1}{n}\sum_{k=1}^{n} \underbrace{\prod_{j=0}^{k-1}\left(1-\frac{j}{n}\right)}_{u_k}$$

If $k \geq n^{3/5}$ then $u_k \leq \exp\left\{-\frac{k(k-1)}{2n}\right\} \leq e^{-\frac{1}{3}n^{1/5}}$.

If $k < n^{3/5}$ then $u_k = \exp\left\{-\frac{k^2}{2n} + O\left(\frac{k^3}{n^2}\right)\right\}$

So

$$\Pr(G_f \text{ is connected}) =$$

$$\frac{1+o(1)}{n} \sum_{k=1}^{n^{3/5}} e^{-k^2/2n} + O\left(n e^{-n^{1/5}/3}\right)$$

$$= \frac{1+o(1)}{n} \int_0^\infty e^{-x^2/2n}\, dx + O\left(n e^{-n^{1/5}/3}\right)$$

$$= \frac{1+o(1)}{\sqrt{n}} \int_0^\infty e^{-y^2/2}\, dy + O\left(n e^{-n^{1/5}/3}\right)$$

$$\sim \sqrt{\frac{\pi}{2n}}\,.$$

Formula for $T(n,k)$:

$$T(n,1) = n^{n-2}$$ Cayley's Formula

$$T(n,k) = \sum_{l=0}^{n-k} \binom{n-k}{l} (l+1)^{l-1}\, T(n-l-1, k-1)$$

#vertices in k'th tree is $l+1$

$$= \sum_{l=0}^{n-k} \binom{n-k}{l} (l+1)^{l-1} (k-1)(n-l-1)^{n-k-l-1}$$ induction

Abel's Formula

$$\sum_{l=0}^{m} \binom{m}{l} (x+l)^{l-1} (y+m-l)^{m-l-1} = \left(\frac{1}{x}+\frac{1}{y}\right)(x+y+m)^{m-1}$$

Take $m=n-k$, $x=1$, $y=k-1$.

$\square$

Number of cycles:

Let Z_k = # of cycles of length k.

$$E(Z_k) = \binom{n}{k}(k-1)!\, n^{-k} = \frac{1}{k}\prod_{j=0}^{k-1}\left(1 - j/n\right)$$

If $Z = Z_1 + \cdots + Z_n$ then

$$E(Z) = \sum_{k=1}^{n} \frac{1}{k}\prod_{j=0}^{k-1}\left(1 - j/n\right)$$

$$\sim \int_1^\infty \frac{1}{x} e^{-x^2/2n}\, dx$$

$$\sim \log_e n .$$

Number of vertices on cycles:

$$E\left(\sum_{k=1}^{n} k Z_k\right) = \sum_{k=1}^{n} \prod_{j=1}^{k-1} \left(1 - j/n\right)$$

$$\sim \sqrt{\frac{\pi n}{2}}\,.$$