Random Mappings

Let \( f \) be chosen uniformly at random from the set of all \( n \) mappings from \([n] \rightarrow [n]\).

Let \( D_f \) be the digraph \( ([n], (x, f(x))) \)

and let \( G_f \) be obtained from \( D_f \) by ignoring orientation.
Ex: \( \alpha \{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \} \)

\( \beta(\alpha) = \{ 2, 10, 5, 4, 2, 5, 8, 9, 7, 5 \} \)

In general, \( D_\alpha \) consists of one cyclic component, where each such component consists of a directed cycle \( C \) with lines rooted at each vertex of \( C \).
Thm 1

\[ \Pr(G_f \text{ is connected}) = \sqrt{\frac{\pi}{2n}} \]

**Proof**

Let \( T(n, k) \) denote the number of forests with vertex set \([n]\), \( k \) blocks, in which \( 1, 2, \ldots, k \) are in different blocks. We show later that

\[ T(n, k) = k^n n^{-k-1} \]
\[ P_r(G_f \text{ is connected}) = \]

\[ n^{-n} \sum_{k=1}^{n} \binom{n}{k} \frac{1}{(k-1)!} \Gamma(n, k) \]

\[ \text{choose cycle of length } k \]

\[ = \frac{1}{n} \sum_{k=1}^{n} \prod_{j=0}^{k-1} \left( 1 - \frac{j}{n} \right) \]

\[ = u_{k}^{n} \]

\[ \text{if } k \geq n^{3/5} \text{ then } u_{k} \leq \exp \left\{ -\frac{k(k-1)}{2an} \right\} \leq e^{-\frac{1}{3} n^{4/5}}. \]

\[ \text{if } k < n^{3/5} \text{ then } u_{k} = \exp \left\{ -\frac{k^2}{2an} + O \left( \frac{k^3}{n^2} \right) \right\} \]
So

\[
P_f( G_f \text{ is connected} ) = \frac{1+o(1)}{n} \sum_{k=1}^{n^{1/5}} e^{-k^2/2n} + O(n e^{-n^{1/5}/3})
\]

\[
= \frac{1+o(1)}{n} \int_0^\infty e^{-x^2/2n} \, dx + O(n e^{-n^{1/5}/3})
\]

\[
= \frac{1+o(1)}{\sqrt{n}} \int_0^\infty e^{-y^2/2} \, dy + O(n e^{-n^{1/5}/3})
\]

\[
\approx \sqrt{\frac{\pi}{2n}}.
\]
Formula for $T(n,k)$:

$$T(n,k) = n^{n-2} \quad \text{Cayley's Formula}$$

$$T(n,k) = \sum_{l=0}^{n-k} \binom{n-k}{l} (l+1)^{l-1} T(n-l-1, k-1)$$

**Abel's Formula**

$$\sum_{c=0}^{m} \binom{m}{c} (xc + l)^{l-1} (y + m - c)^{m-l-1} = (\frac{1}{x} + \frac{1}{y})(xc + y + m)^{m-1}$$

Take $m = n-k$, $x = 1$, $y = k-1$. 

\[ \square \]
Number of cycles:

Let $Z_k = \#\text{cycles of length } k$.

$$E(Z_k) = \binom{n}{k}(k-1)! \cdot n^{-k} = \frac{1}{k} \prod_{j=0}^{k-1} (1 - \frac{j}{n})$$

If $Z = Z_1 + \cdots + Z_n$ then

$$E(Z) = \sum_{k=1}^{n} \frac{1}{k} \prod_{j=0}^{k-1} (1 - \frac{j}{n})$$

$$\sim \int_{-\infty}^{\infty} \frac{1}{x} e^{-x^2/2n} \, dx$$

$$\sim \log n.$$
Number of vertices on cycles:

\[ E \left( \sum_{k=1}^{n} k Z_k \right) = \sum_{k=1}^{n} \prod_{j=1}^{k-1} (1 - \frac{j}{n}) \]

\[ \sim \sqrt{\frac{\pi n}{2}}. \]