Perfect Matchings in Bipartite 2-OUT

$B_{k\text{-out}}$ is a random bipartite graph with vertex partition $X \cup Y$ where $|X| = |Y| = n$.

Each $x \in X$ chooses k random nbrs in Y

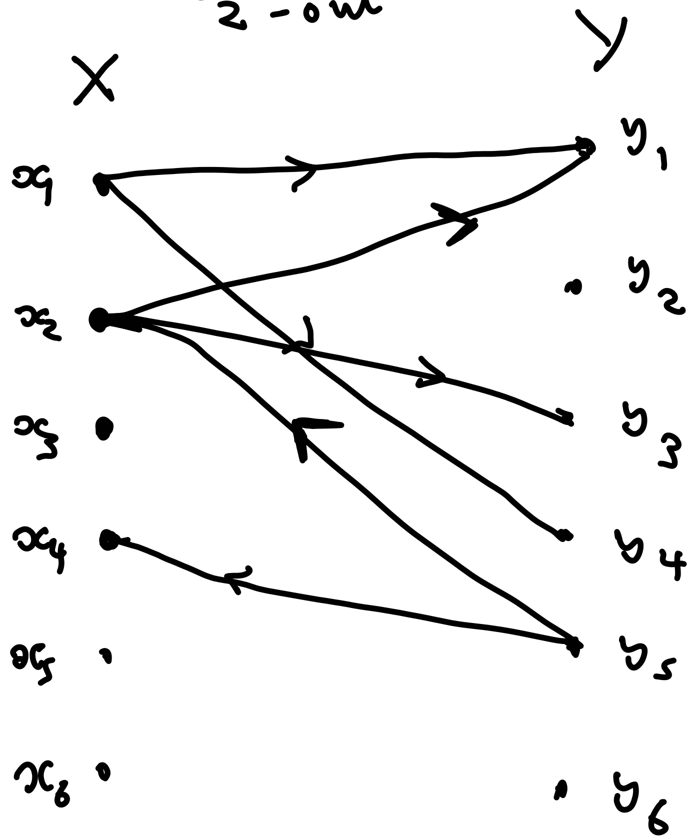
Each $y \in Y$ chooses k random nbrs in X.

Theorem

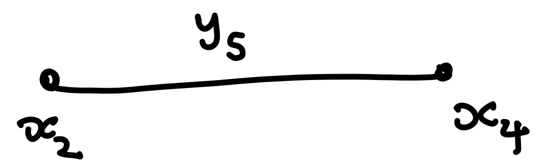
$B_{2\text{-out}}$ has a perfect matching whp.

Algorithmic Proof

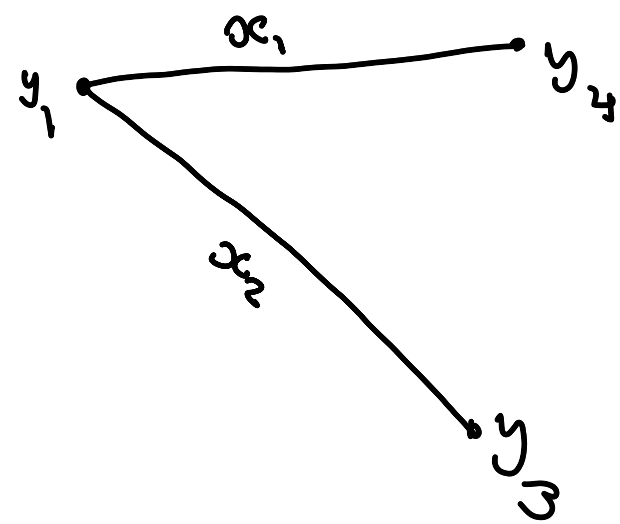
B_2-out



G_1: n random edges

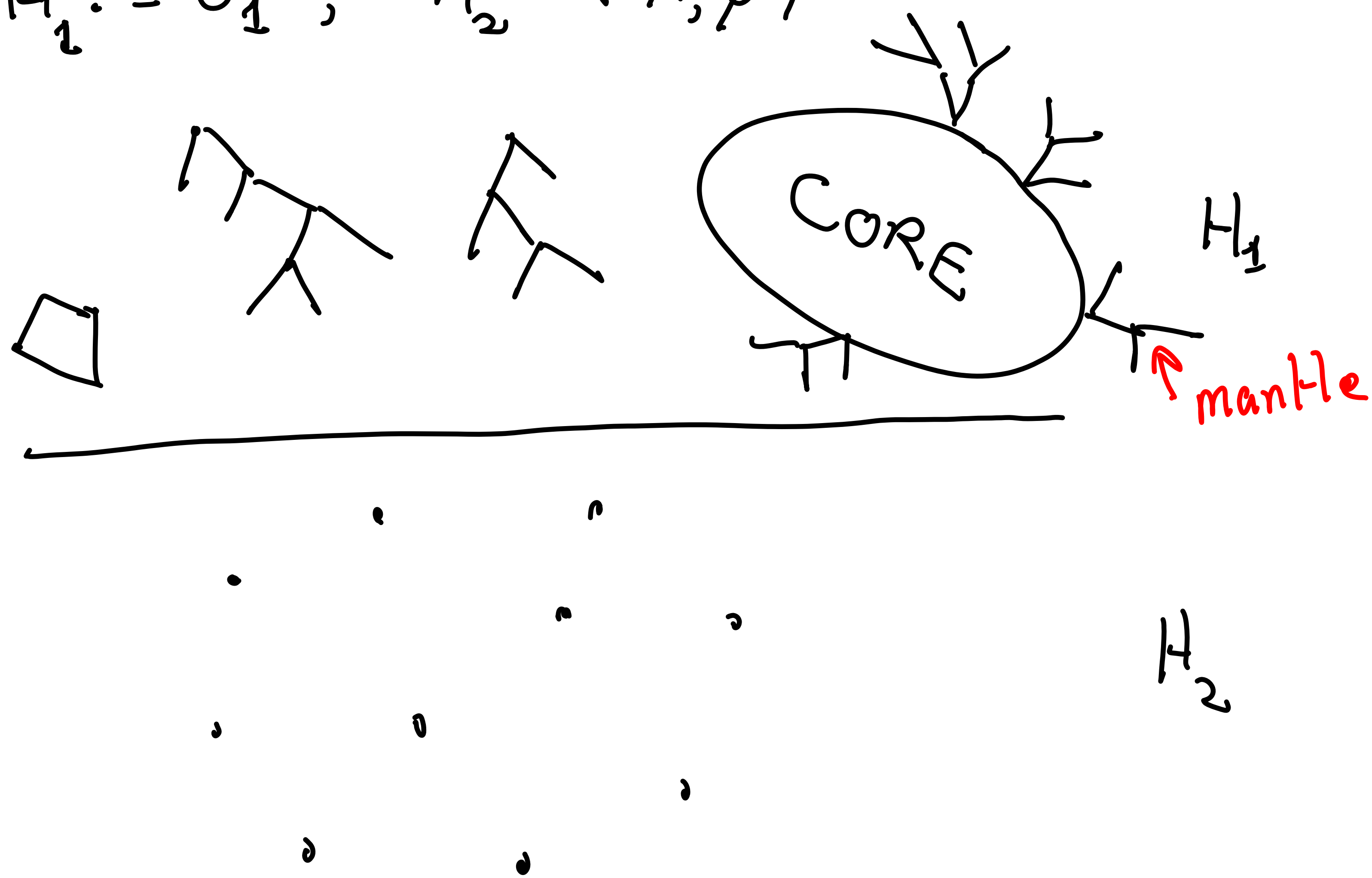


G_2: n random edges

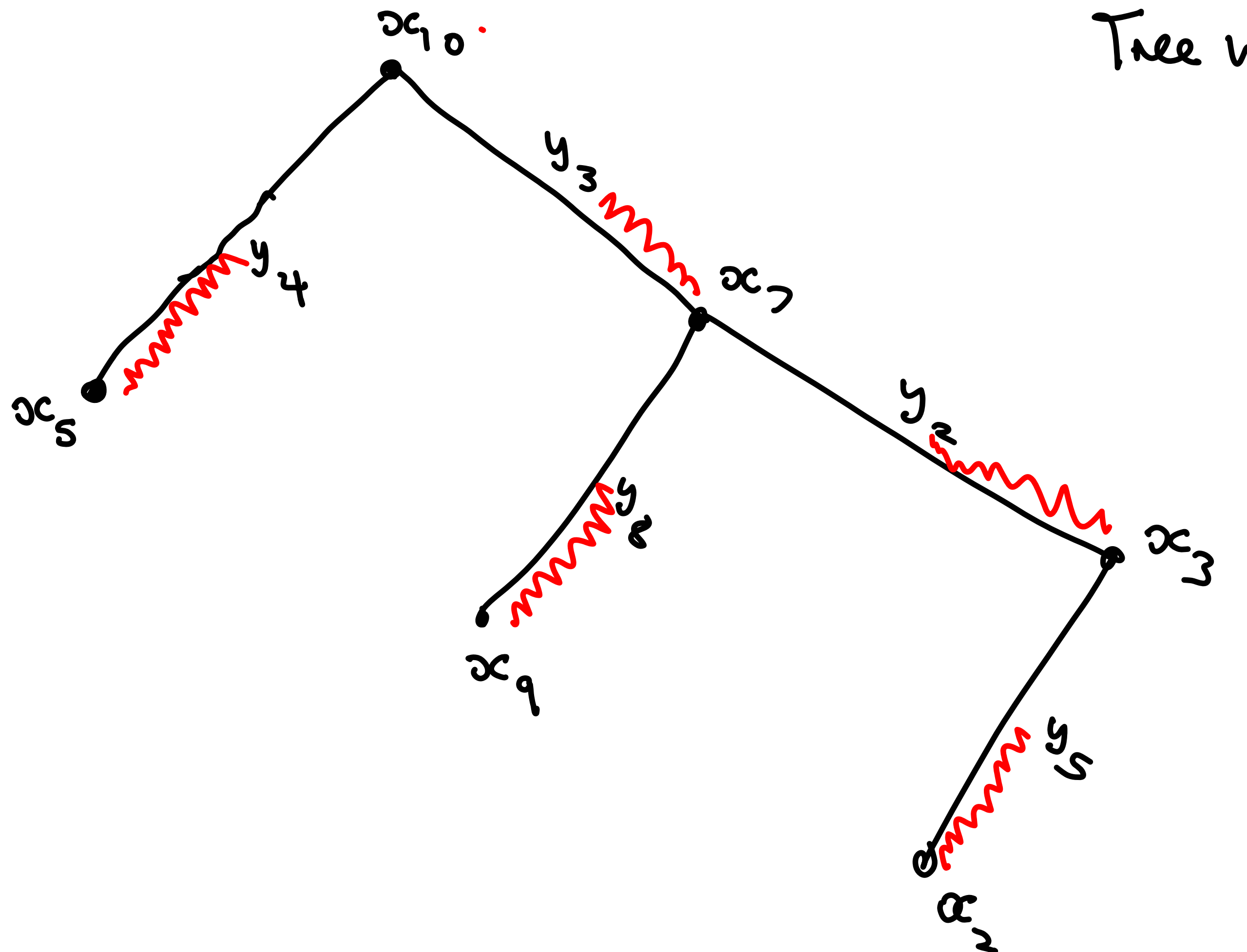


Algorithm

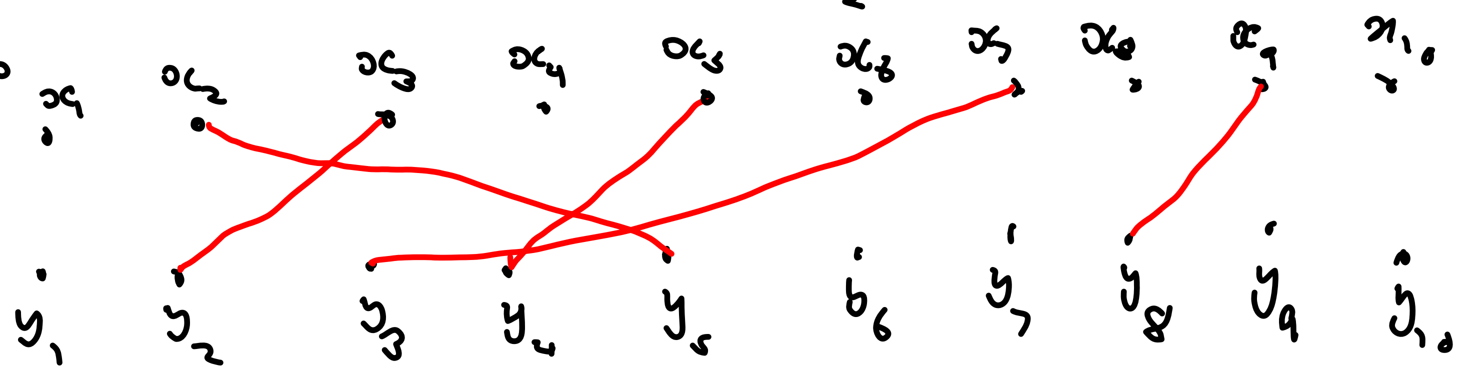
$H_1 := G_1$; $H_2 := (X, \emptyset)$

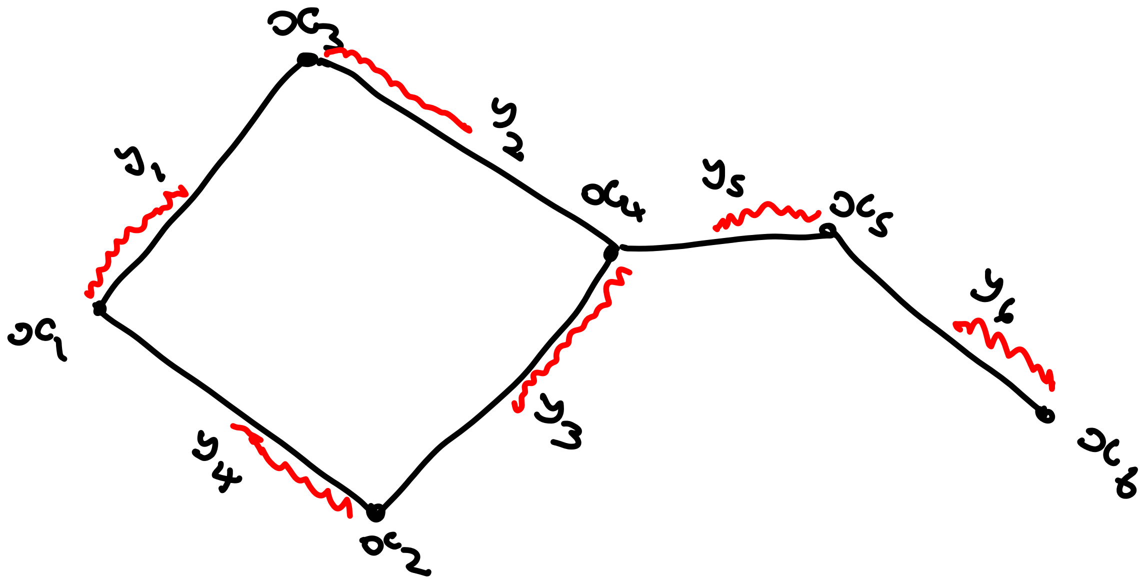


Tree in H_1

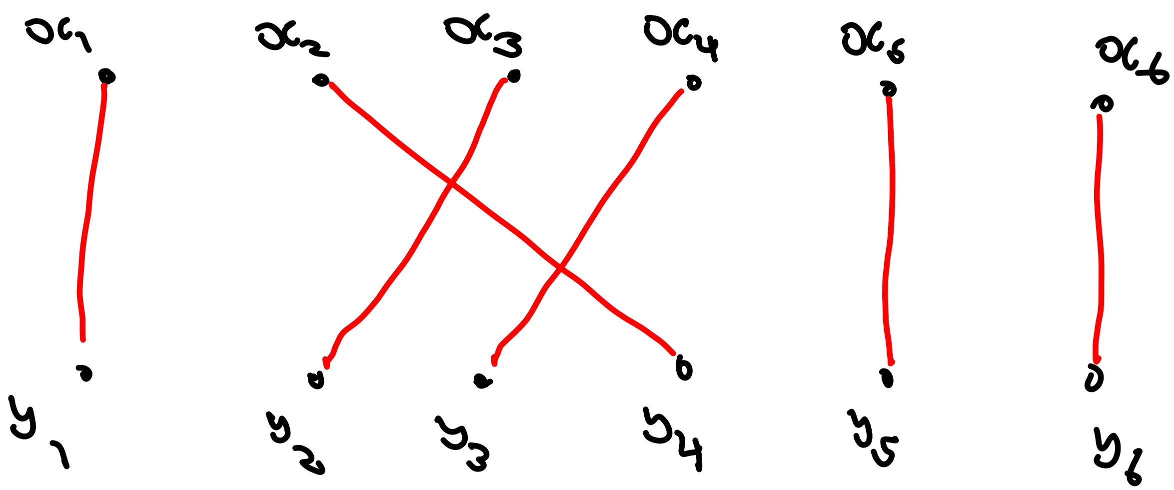


Yields





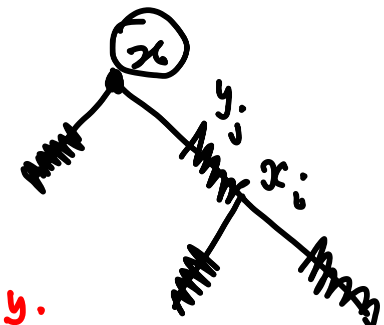
x3
y2
y1
x4
y5
x5
y6
x1
y3
y4
x6
x2



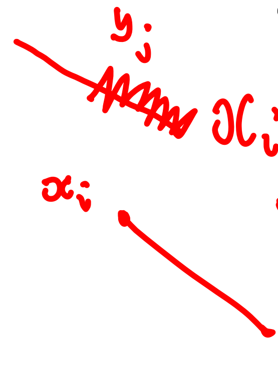
x1
x2
x3
x4
x5
x6
y1
y2
y3
y4
y5
y6

Step 1: If every isolated tree of H_1 contains a marked vertex: FOUND PERFECT MATCHING.

Step 2: Choose unmarked isolated tree T_j;
Choose root x for T_j;
Mark x.

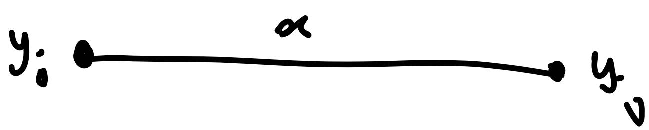


Step 3: Add edge with label x to H_2



yields

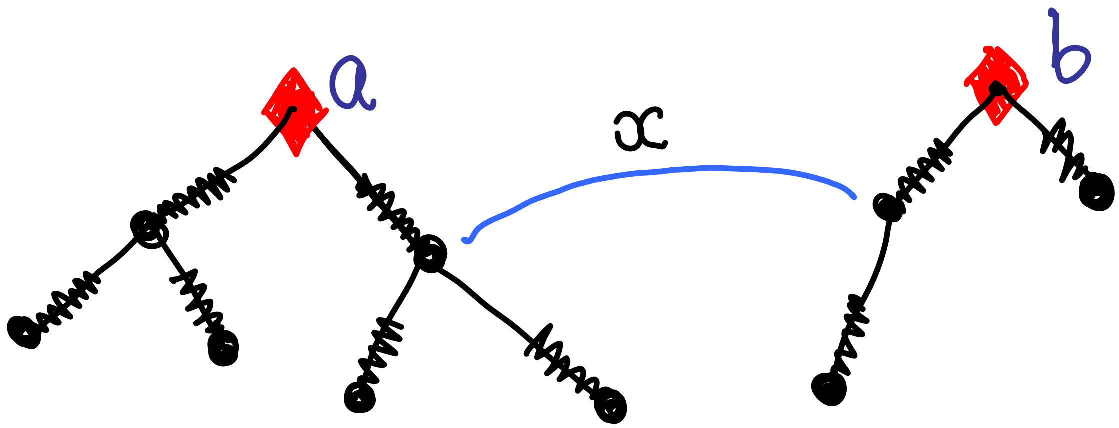
in B_2

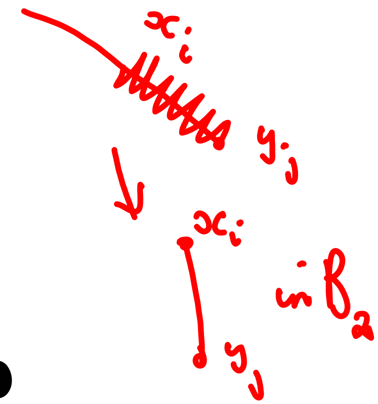


x chose y_i, y_j

Step 4 : Possibilities.

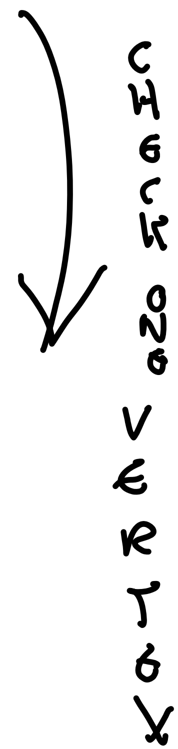
(1)

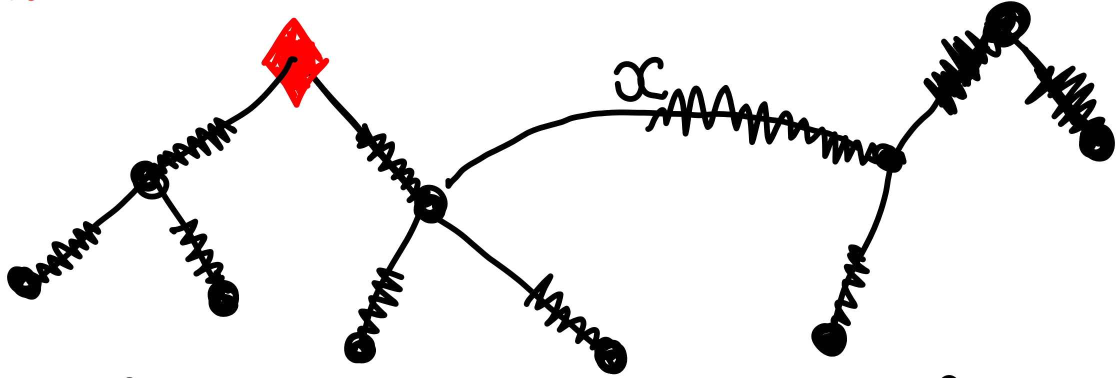




● - checked vertex

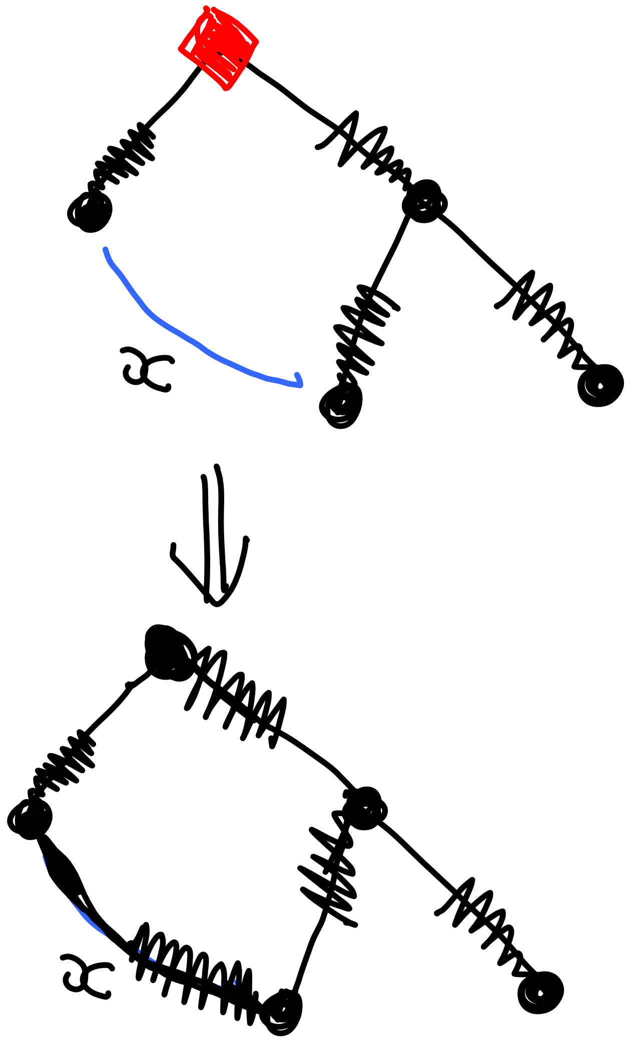
◆ - unchecked vertex



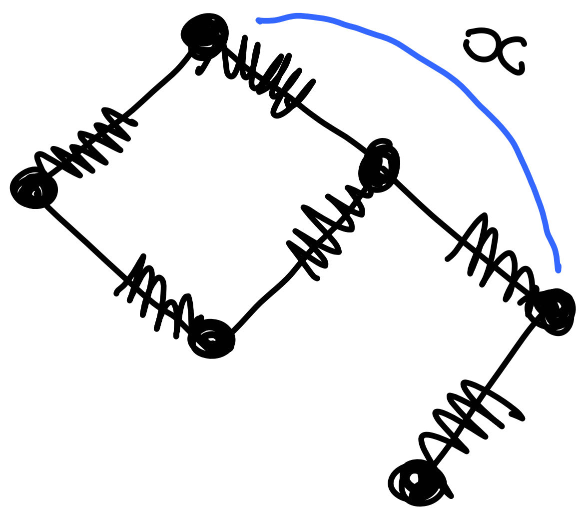


Give preference to vertex b if edge b in CORE of H_1 and edge a is not.

(ii)



(iii)



FAIL

In cases (i) & (ii) delete edge x from H_1.

Repeat from Step 1

Invariants

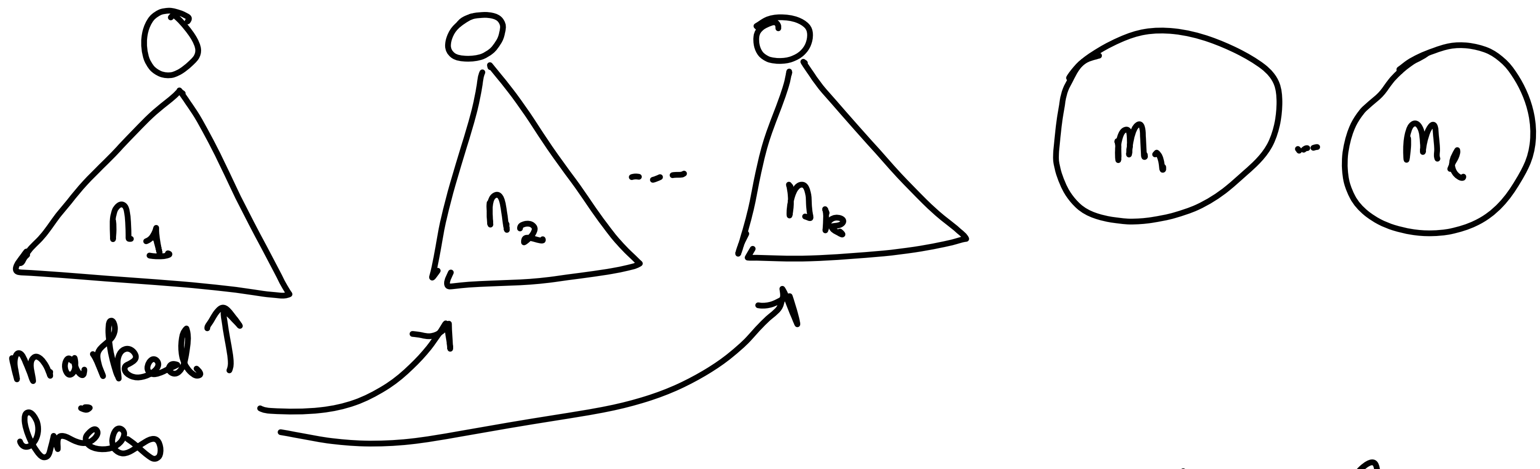
(i) # marked vertices $= n -$ # edges in H_1.

Each round marks one vertex and deletes one edge of H_1.

(ii) # checked vertices $=$ # edges of H_2.

Each round checks one vertex and adds one edge to H_2.

Suppose there are no unmarked trees.



$\#$ vertices $\quad n_1 + n_2 + \cdots + n_k + m_1 + \cdots + m_l = n$

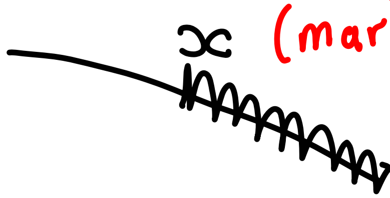
$\#$ edges $\quad (n_1 - 1) + (n_2 - 1) + \cdots + (n_k - 1) + (\geq m_1) + \cdots + (\geq m_l) = n - k$

$\Rightarrow$ m_1 - m_l are unicyclic

The edges y—x give a matching M_1 in B_2 of size $n-k$ covering all unmarked OC's.

Note $\#$rounds $=$ $\#$edges lost $= k$

H_2 contains k edges, also yielding matching M_2 in B_2.

 (marked in H_1)

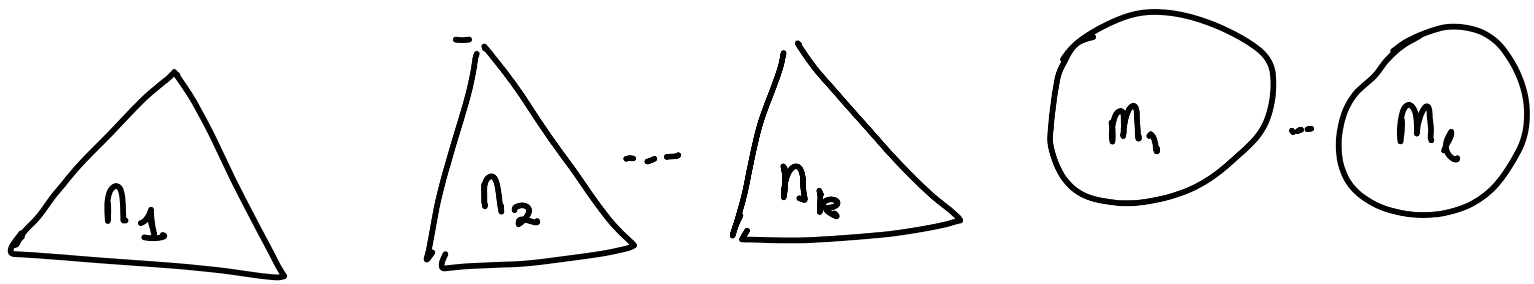
The  derived edge covers x.

M_1 does not cover x, but M_2 does.

Finally, suppose that M_1 does not cover y i.e. $\nexists$ . This edge was deleted and so y is a checked vertex of H_2 and is covered by M_2.

Thus $M_1 \cup M_2 = n$ edges covering $X \cup Y$, i.e. perfect matching.

Conversely, suppose H_1 consists of trees and unicyclic components



$$\# \text{edges} = n - k$$
$$= n - \#\text{marked vertices}$$

So every tree has a marked vertex and algorithm stops as soon as this happens.

Probability of Failure

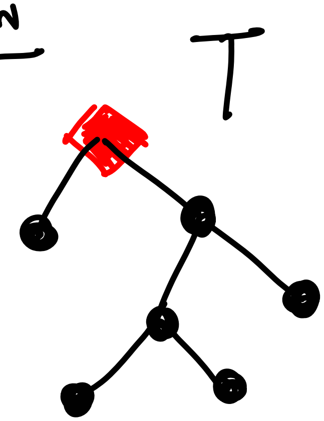
Claim (proved below)

Whp H_1 consists only of trees and unicyclic components before $.49n$ rounds.

Assume claim: H_2 consists of $\leq .49n$ random edges and so whp only contains trees and unicyclic components and so case (iii) of Step 4 does not happen.

Proof of Claim

Each H_2-tree has one unchecked vertex



Edge of H_1 corr. to unchecked vertices of H_2.

If ■ corresponds to edge of CORE then our rule ⇒ every vertex of T corresponds to edge of CORE.

So # vertices left in (what was) CORE = # trees of H_2 where every vertex corr. to an edge of CORE.

Size of CORE

Suppose $xe^{-x} = 2e^{-2}$, $0 < x < 1$.

Then CORE has $\approx (1 - x/2)^2 n$ edges.

$$.4 \leq x \leq .41$$

$$.63 \leq \left(1 - \frac{x}{2}\right)^2 \leq .64$$

Let Z = # trees in H_2 made up of vertices $y \in Y$ whose edge in H_1 belongs to CORE.

$$E(Z) \leq o(1) + \sum_{k=1}^{(\log n)^2} \binom{n}{k} k^{k-2} \binom{.49n}{k-1} (k-1)! \left(\frac{1}{\binom{n}{2}}\right)^{k-1} \cdot (.64)^{k-1} \cdot \left(1 - \frac{k(n-k)}{\binom{n}{2}}\right)^{.49n}$$

$$\leq o(n) + .64n \sum_{k=1}^{(\log n)^2} \frac{k^{k-2}}{k!} (.64)^{k-1} \exp\left\{-\frac{.98k(n-k)}{(n-1)}\right\}$$

$$\leq .64n \sum_{k=1}^{(\log n)^2} \frac{k^{k-2}}{k!} (.64)^{k-1} \exp\left\{- \frac{.98k(n-k)}{(n-1)}\right\} + o(n)$$

$$\leq (1+o(1))(.64)n \left(e^{-.98} \left(\frac{.6}{2} + \frac{3(.64)^3}{4} + \frac{16(.64)^3}{24} \right) + \sum_{k=5}^{(\log n)^2} \frac{1}{k^{5/2}} \cdot \frac{1}{.64} \cdot (.64e^{.02})^k \right)$$

$\leq \frac{1}{3}$

$$\leq \frac{1}{5^{5/2} \times .64 \times (1-.65 \times e^{.02})}$$

$$\leq \frac{1}{15}$$

So, after .49 rounds,
in expectation,
edges left in CORE,
is $\leq \frac{2}{5}$ of original,
and Chebyshev can be used to show this whp.

But deleting $\approx \frac{3}{5}$ of CORE's edges will whp leave just trees and unicyclic components:

Choose n random edges.

Build CORE

Delete $\approx \frac{3}{5}$ of edges.

(i) Whp $\approx \frac{3}{5}$ of edges of CORE are deleted.

(ii) Graph has $\approx \frac{2}{5}n$ edges and so has only trees plus unicyclic components.

So whp algorithm finishes before $.49n$ rounds with a perfect matching.