Largest component in $G_{n,p}$ near $p = 1/n$.

Theorem

Let $p = \frac{1}{n} + \frac{\lambda}{n^{4/3}}$ where $|\lambda| = O(1)$.

Let $C_1, C_2, \ldots$ denote the connected components of $G_{n,p}$ where $|C_1| \geq |C_2| \geq \cdots$. Then

(i) $$E\left(\sum_j |C_j|^2\right) \leq \begin{cases} 3n^{4/3} & \lambda = 0 \\ 4n^{4/3} & 0 < |\lambda| \leq 1/10 \\ n^{4/3}\left[2 + 5|\lambda|^{1/2}\right]. & |\lambda| \geq 1/10 \end{cases}$$

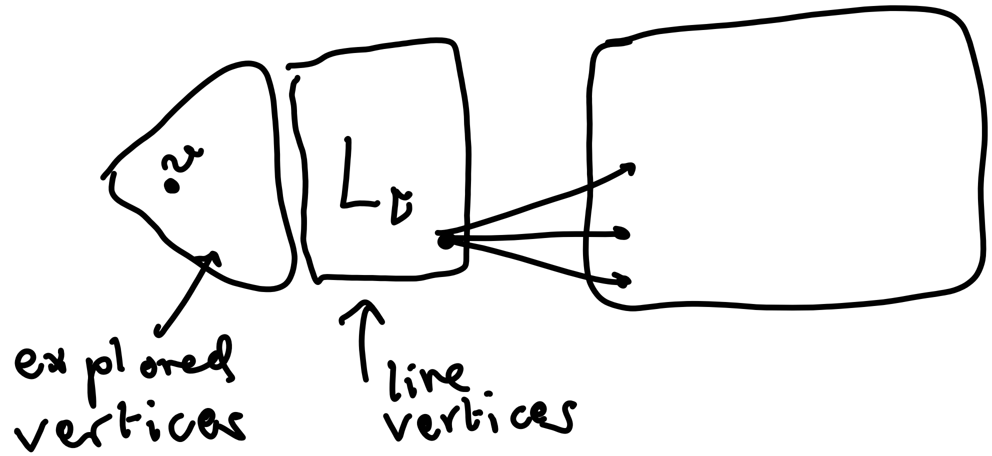
$\Downarrow$ Markov

(ii) $$\Pr(|C_1| \geq An^{2/3}) \leq A^{-2}\left(4 + 5\sqrt{|\lambda|}\right).$$

(iii) $$\Pr(|C_1| \leq \delta n^{2/3}) \leq (33 + 21|\lambda|)\,\delta^{8/5}.$$

if δ is sufficiently small and n sufficiently large.

Fix vertex v. In BFS from v we construct sequence of sets



$Y_t = |L_t|$

$Y_0 = 1$

$$Y_t = \begin{cases} Y_{t-1} + \eta_t - 1, & Y_{t-1} > 0 \\ \eta_t & Y_{t-1} = 0 \end{cases}$$

where $\eta_t = B(n - Y_{t-1} - 1, p)$.

$\eta_1, \eta_2, \ldots$ are independent.

Note that if $C(v)$ is the component containing v then

$$|C(v)| = \min\{t : Y_t = 0\}.$$

$$\overset{d}{=} T.$$

$$S_t = 1 + \sum_{i=1}^{t} (\xi_i - 1)$$

$\xi_1, \xi_2, \dots$ are indep. copies of $B(n, p)$.

We couple so that $\eta_1 \leq \xi_1$, $\eta_2 \leq \xi_2$, ...

It follows that

$$S_t \geq Y_t \quad \text{for} \quad t = 0, 1, 2, \dots$$

$$E(S_{t+1} - S_t \mid S_t) = np - 1.$$

Let

$$\hat{S}_t = S_t - t\,|np-1|$$

Then

$$E(\hat{S}_{t+1} \mid \hat{S}_t) = (np-1) - |np-1| \leq 0$$

and so $(\hat{S}_t)$ is a super-martingale.

Now fix an integer $H > 0$ and let

$$\gamma = \min\{t \geq 1 : S_t \geq H \text{ or } S_t = 0\}$$

Note that

$$S_\gamma \geq H \Rightarrow Y_\gamma \leq S_\gamma$$

Let $\tau_0 = \min\{t \geq 0 : Y_{\gamma+t} = 0\}$

$$\tau \leq \gamma + \tau_0 \mathbb{1}_{\{S_\gamma \geq H\}}$$

$$[\, S_\gamma = 0 \Rightarrow \tau \leq \gamma \,].$$

$$E(T) \leq E(\gamma) + E(T_0 \mid S_\gamma \geq H)\, P(S_\gamma \geq H)$$

We prove

(i) $$P(S_\gamma \geq H) \leq \frac{1 + E(\gamma)\,|np-1|}{H}$$

(ii) $$E(\gamma) \leq \frac{H+2}{npq - 4H\,|np-1|}$$

We make sure denominator is positive.

(iii) $$E(T_0 \mid S_\gamma \geq H) \leq \left(\frac{2(H+np)}{p}\right)^2$$

So,

$$E(T) \leq$$

$$\frac{H+2}{npq - 4H|np-1|} + \left(\frac{2(H+np)}{p}\right)^{\frac{1}{2}} \left(\frac{npq - 3H|np-1|}{npq - 4H|np-1|}\right) \cdot \frac{1}{H}$$

We choose H to (approximately) minimise the RHS.

If $\lambda = 0$

$$E(T) \leq \frac{H+2}{n-1} + \frac{\sqrt{2n(H+1)}}{H}$$

Put $H = n^{1/3} \Rightarrow E(T) \leq 3n^{1/3}$.

If $0 < |\lambda| < \frac{1}{10}$ then

$$E(\tau) \leq 2(H+2) + \frac{\sqrt{(2+o(1))n(H+1)} \times 7}{6H}$$

Putting $H = n^{1/3}$ gives

$$E(\tau) \leq 4n^{1/3}.$$

If $|\lambda| \geq \frac{1}{10}$ we put $H = \frac{n^{1/3}}{10|\lambda|}$ and then

$$E(\tau) \leq 2H + \frac{\sqrt{(2+o(1))nH} \times 7}{6H}$$

$$\leq n^{1/3}\left[2 + 5|\lambda|^{1/2}\right].$$

Now write

$$E(T) = E(|C(v)|)$$

$$= \frac{1}{n} \sum_{v=1}^{n} E(|C(v)|)$$

$$= \frac{1}{n} E\Big(\sum_j |C_j|^2\Big)$$

So $$E\Big(\sum_j |C_j|^2\Big) \leq n\, E(T).$$

Main tool [OPTIONAL STOPPING]

Let $Z_0, Z_1, \dots Z_t, \dots$ be a random process.

T is a stopping time if the event $\{T = t\}$ depends only on $Z_0, Z_1, \dots, Z_t$ and not on the future.

Optional Stopping

Suppose T is a stopping time.

(i) (Z_t) is a martingale $\Rightarrow E(Z_T) = E(Z_0)$.

(ii) (Z_t) is a supermartingale $\Rightarrow E(Z_T) \leq E(Z_0)$.

(iii) (Z_t) is a submartingale $\Rightarrow E(Z_T) \geq E(Z_0)$

We must also assume (Z_t) is bounded.

$$1 = E(\hat{S}_0) \geq E(\hat{S}_\gamma) = E(S_\gamma) - E(\gamma)(np-1)^+$$

$$\geq H\, P(S_\gamma \geq H) - E(\gamma)(np-1)^+$$

so

$$P(S_\gamma \geq H) \leq \frac{1 + E(\gamma)|np-1|}{H}$$

Lemma

Given $S_\gamma \geq H$, the conditional distribution of $S_\gamma - H \overset{d}{\leq} B(n,p)$.

Proof

$\xi = B(n,p) = I_1 + I_2 + \cdots + I_n$

Given $\xi \geq r$, $\xi - r \overset{d}{\leq} B(n,p)$.*

$\left\{ \begin{array}{l} \text{Suppose } r = \sum_{j=1}^{J} I_j \text{ so that } \xi - r \text{ has distribution} \\ \overset{d}{\leq} B(n-J, p). \end{array} \right.$

Conditioned on $\{\gamma = \ell\} \cap \{S_{\ell-1} = H - r\} \cap \{S_\gamma \geq H\}$,

$$S_\gamma - H \overset{d}{=} \xi_\ell - r \overset{d}{\leq} B(n,p).$$

Now average over ℓ, r.

$\square$

* $A \overset{d}{\leq} B$ if $\Pr(A \geq x) \leq \Pr(B \geq x)$, $\forall x$.

Write

$$S_\gamma^2 = H^2 + 2H(S_\gamma - H) + (S_\gamma - H)^2$$

Then, lemma on p10 implies

$$E(S_\gamma^2 \mid S_\gamma \geq H) \leq H^2 + 2Hnp + npq + (np)^2$$

$$\leq H^2 + 3H.$$

Define

$$t \wedge \gamma = \min\{t, \gamma\}.$$

and

$$A_t = S^2_{t\wedge\gamma} - B\,(t\wedge\gamma)$$

where

$$B = npq - 2H|1-np|.$$

We claim that

(A_t) is a sub-martingale

$$E\left(S_{t+1}^2 - S_t^2 \mid S_t\right) =$$

$$2E\left(S_t(\xi_{t+1}-1)\right) + E\left((\xi_{t+1}-1)^2\right)$$

$$= 2S_t(np-1) + npq + 1 - np$$

$$\geq \underbrace{npq - 2H|np-1|}_{B}, \quad \forall\ t \leq \gamma.$$

$$E\left(\left[S_{t+1}^2 - B(t+1)\right] - \left[S_t^2 - Bt\right] \,\middle|\, S_t\right) \leq 0, \quad t \leq \gamma.$$

So

$$A_0 \leq E(A_\gamma)$$

or

$$1 \leq E(S_\gamma^2) - BE(\gamma)$$

So

$$1 + BE(\gamma) \leq E(S_\gamma^2) = E(S_\gamma^2 \mid S_\gamma \geq H) P_r(S_\gamma \geq H)$$

$$\leq (H+3)\left(1 + E(\gamma)\,|np-1|\right)$$

So

$$E(\gamma) \leq \frac{H+2}{B-(H+3)\,|np-1|} \leq \frac{H+2}{npq - 4H\,|np-1|}$$

We ensure this is positive.

Now consider

$\tau_0 = \min\{t \geq 0 : Y_{\gamma+t} = 0\}$

$$Z_t = Y_{\gamma + t\wedge\tau_0} + \sum_{j=1}^{t\wedge\tau_0} jp$$

If $t < \tau_0$ then

$\sigma = (t+1)\wedge\tau_0$

$$E(Z_{t+1} - Z_t \mid Z_t) = E(\eta_{\gamma+\sigma} + \sigma p)$$

$$= -1 + (n - Y_{\gamma+t\wedge\tau_0} - (\gamma + t\wedge\tau_0) + \sigma)p$$

$$\leq 0$$

and $Z_{t+1} = Z_t$ if $t \geq \tau_0$.

So (Z_t) is a supermartingale.

$$H + np \geq E(S_\gamma \mid S_\gamma \geq H)$$ Lemma on p10

$$\geq E(Z_0 \mid S_\gamma \geq H)$$ $S_\gamma \geq Y_\gamma$

$$\geq E(Z_{\tau_0} \mid S_\gamma \geq H)$$ Optional Stopping

$$\geq E(\tau_0^2 \mid S_\gamma \geq H)\, p/2$$ Take sum only.

By Cauchy-Schwarz

$$E(\tau_0 \mid S_\gamma \geq H) \leq E(\tau_0^2 \mid S_\gamma \geq H)$$
$$\leq \left(\frac{2(H+np)}{p}\right)^{1/2}$$

Proof of (iii)

Fix $h = An^{1/3}$, $A = O(1)$ to be determined.

Stage 1

$$T_h = \begin{cases} \min\{t \le \frac{n}{8h} : Y_t \ge h\} \\ \frac{n}{8h} \end{cases}$$
← set non-empty

otherwise

If $Y_{t-1} > 0$ then

$$Y_t^2 - Y_{t-1}^2 = (\eta_t - 1)^2 + 2(\eta_t - 1) Y_{t-1}.$$

If $Y_{t-1} \le h$ then

$$E(Y_t^2 - Y_{t-1}^2 \mid Y_{t-1}) \ge (n - t - h) pq - 2(t+h) p h.$$

$$\ge \frac{1}{2}.$$

If $Y_{t-1} = 0$ then $E(Y_t^2 - Y_{t-1}^2) = E(\eta_t^2) \geq \frac{1}{2}$, under these assumptions.

So $Y^2_{t \wedge \tau_h} - \frac{1}{2}(t \wedge \tau_h)$ is a submartingale

and so

$$E(Y^2_{\tau_h}) - \frac{1}{2}\tau_h \geq 0.$$

Lemma on P13 $\Rightarrow$

$$E(Y^2_{\tau_h}) \leq h^2 + 3h \leq 2h^2.$$

So

$$2h^2 \geq E(Y^2_{\tau_h}) \geq \frac{1}{2}E(\tau_h) \geq \frac{T_1}{2}P_r\left(T_h = \frac{n}{8h}\right)$$

or

$$P_r\left(\tau_h = \frac{n}{8h}\right) \leq \frac{32h^3}{n}.$$

$$\tau_0 = \begin{cases} \min\{t \le \delta n^{2/3} : Y_{\tau_h + t} = 0\} & \leftarrow \text{set non-empty} \\ \delta n^{2/3} & \text{otherwise} \end{cases}$$

$$M_t = h - \min\{h, Y_{\tau_h + t}\}.$$

If $\quad 0 < M_{t-1} < h \quad$ then

$$M_t^2 - M_{t-1}^2 = (\eta_{\tau_h + t} - 1)^2 + 2(1 - \eta_{\tau_h + t}) M_{t-1}$$

and so

$$E(M_t^2 - M_{t-1}^2 \mid M_{t-1}) \le npq + 2h\left(1 - \left(n - \frac{n}{8h} - \delta n^{2/3}\right)p\right)$$

$$\le 2(1 + A|\lambda|).$$

If $Y_{t-1} \geq h$ then $M_{t-1} = 0$ and $M_t \leq 1$.

So $Z_t = M^2_{t \wedge T_0} - 2(1 + A|\lambda|)(t \wedge T_0)$ is a supermartingale.

Now use P_h, E_h to denote conditioning on $\{Y_{\rho_h} \geq h\}$.

$Z_0 = 0$ and so

$$0 \geq E(Z_{T_0}) = E_h(M^2_{T_0}) - 2(1+A|\lambda|)E(T_0)$$

$$\geq E_h(M^2_{T_0}) - (1+A|\lambda|)\delta n^{2/3}.$$

So

$$P_h(T_0 < \delta n^{2/3}) \leq P_h(M_{T_0} \geq h) \leq \frac{E_h(M^2_{T_0})}{h^2} \leq \frac{(1+A|\lambda|)\delta n^{2/3}}{h^2}$$

implies

So

$$P(T_0 < \delta n^{2/3}) \leq P\left(T_h = \frac{n}{8h}\right) + P_h(T_0 < n^{2/3})$$

$$\leq \frac{32h^3}{n} + \frac{(1+A|\lambda|)\cdot \delta n^{2/3}}{h^2}$$

or

$$P(T_0 < \delta n^{2/3}) \leq 32A^3 + \frac{(1+A|\lambda|)\,\delta}{A^2}.$$

Putting $A = \delta^{1/5}$ for simplicity, we get

which gives

$$P(T_0 < \delta n^{2/3}) \leq (33 + 2|\lambda|)\,\delta^{3/5}.$$

We finally note that

$$|C_1| < \delta n^{2/3} \implies |C(v)| < \delta n^{2/3}$$
$$\implies T < \delta n^{2/3}$$
$$\implies T_0 < \delta n^{2/3}.$$