Preferential Attachment.

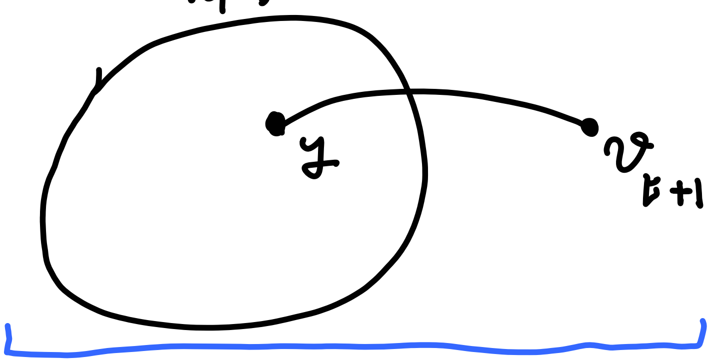
Fix $m > 0$, constant.

Sequence of graphs

$\Gamma_1, \Gamma_2, \ldots, \Gamma_{m-1}, G_1, \Gamma_{m+1}, \Gamma_{m+2}, \ldots \Gamma_{2m-1}, G_2, \ldots \Gamma_{mt-1}, G_t, \ldots$

$\Gamma_{(m-1)t+1}, \ldots \Gamma_{mt-1}, G_t$ have vertex set $v_1, v_2, \ldots, v_t$

Γ_τ, $\in [mt, mt+m-1]$



$\Gamma_{\tau+1}$

$$Pr(y = v) = \begin{cases} \dfrac{\deg(v, \Gamma_\tau)}{2\tau+1} & v \neq v_{t+1} \\[2ex] \dfrac{1}{2\tau+1} & v = v_{t+1} \end{cases}$$

Expected Degree Sequence.

$D_k(t) = \#$ of vertices of degree k in G_t,
$m \le k = \tilde{O}(t^{1/2})$.

$\bar{D}_k(t) = E(D_k(t))$.

$$E(D_k(t+1) \mid G_t) = D_k(t) + 1_{k=m} + \epsilon(k,t)$$
$$+ m\left(\frac{(k-1)D_{k-1}(t)}{2mt} - \frac{kD_k(t)}{2mt}\right)$$

$$|\epsilon(k,t)| = O\left(\sum_{i=2}^{m} \frac{(k-i)^i D_{k-i}(t)}{(mt)^i}\right) = O\left(\frac{k}{t}\right) = \tilde{O}(t^{-1/2}).$$

$\ell D_\ell(t) \le 2mt$

To account for multiple edges and denominator being $2mt + (\le m)$.

Taking expectations over G_t,

$$\bar{D}_k(t+1) = \bar{D}_k(t) + 1_{k=m} + \tilde{O}(t^{-1/2})$$

$$+ m\left(\frac{(k-1)\bar{D}_{k-1}(t)}{2mt} - \frac{k\bar{D}_k(t)}{2mt}\right)$$

Under the assumption $\bar{D}_k(t) \sim d_k t$ we are led to the recurrence

$$d_k = 1_{k=m} + \left[(k-1)d_{k-1} - k d_k\right]/2$$

or

$$d_k = \frac{k-1}{k+2} d_{k-1} + \frac{1_{k=m}}{k+2} \times 2 \qquad k \geq m$$

$$= 0 \qquad k < m$$

$$d_k = \frac{k-1}{k+2} d_{k-1} + \frac{1_{k=m}}{k+2} \times 2 \qquad k \geq m$$

$$= 0 \qquad k < m$$

Therefore

$$d_m = \frac{2}{m+2}$$

$$d_k = d_m \prod_{l=m+1}^{k} \frac{l-1}{l+2}$$

$$= \frac{2m(m+1)}{k(k+1)(k+2)} .$$

Theorem

$$|\bar{D}_k(t) - d_k t| = \tilde{O}(t^{1/2})$$

Proof

Let $\Delta_k(t) = \bar{D}_k(t) - d_k t$. Then

$$\Delta_k(t+1) = \frac{k-1}{2t}\Delta_{k-1}(t) + \left(1 - \frac{k}{2t}\right)\Delta_k(t) + \underbrace{\tilde{O}(t^{-1/2})}_{\leq \alpha t^{-1/2}(\log t)^{\beta}}.$$

Now assume inductively on t that

$$|\Delta_k(t)| \leq A t^{1/2} (\log t)^{\beta} \qquad \forall k \geq 0$$

This is trivially true for small t (make A large) and $k < m$.

So

$$|\Delta_k(t+1)| \leq \frac{k-1}{2t}|\Delta_{k-1}(t)| + \left|\left(1-\frac{k}{2t}\right)\Delta_k(t)\right| + \alpha t^{-1/2}(\log t)^{\beta}$$

$$\leq \frac{k-1}{2t} A t^{1/2} (\log t)^{\beta} + \left(1-\frac{k}{2t}\right) A t^{1/2} (\log t)^{\beta} + \alpha t^{-1/2} (\log t)^{\beta}$$

$$\leq (\log t)^{\beta}\left(A t^{1/2} + \alpha t^{-1/2}\right)$$

$$(t+1)^{1/2} = t^{1/2}\left(1+\frac{1}{t}\right)^{1/2} \geq t^{1/2} + \frac{1}{3t^{1/2}} \qquad t \text{ large enough}$$

$$\leq (\log(t+1))^{\beta}\left(A\left[(t+1)^{1/2} - \frac{1}{3t^{1/2}}\right] + \frac{\alpha}{t^{1/2}}\right)$$

$$\leq A(\log(t+1))^{\beta}(t+1)^{1/2}.$$

$\square$

Concentration

$$\Pr\left(|D_k(t) - \bar{D}_k(t)| \geq u\right) \leq 2\exp\left\{-\frac{u^2}{8mt}\right\}.$$

Proof

Let $Y_1, Y_2, \ldots, Y_{mt}$ be the sequence of choices made in the construction of G_t.

$$Z_i = Z_i(Y_1, Y_2, \ldots, Y_i)$$

$$= E(D_k(t) \mid Y_1, Y_2, \ldots, Y_i).$$

Result follows from

$$|Z_i - Z_{i-1}| \leq 4.$$

Fix $y_1, y_2, \ldots, y_i$ and $\hat{y}_i \neq y_i$. We define map

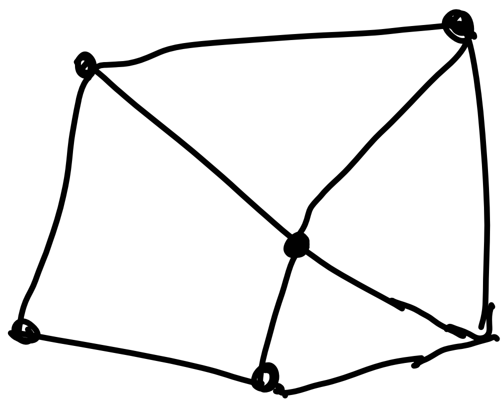
$$y_1, y_2, \ldots, y_{i-1}, y_i, y_{i+1}, \ldots, y_{mb}$$

$$\Big\Downarrow \text{ measure preserving projection } \phi$$

$$y_1, y_2, \ldots, y_{i-1}, \hat{y}_i, \hat{y}_{i+1}, \ldots, \hat{y}_{mb}$$

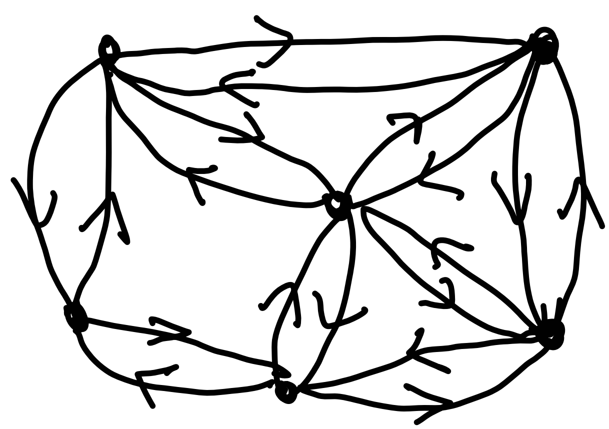
D_k changes by at most 4.

In preferential attachment we can view vertex choices as choice of a random arc

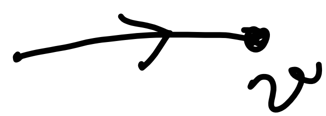


Choose vertex v according to degree

$\Downarrow$



choose random arc



So $Y_1, Y_2, \ldots$ can be viewed as a sequence of arc choices.

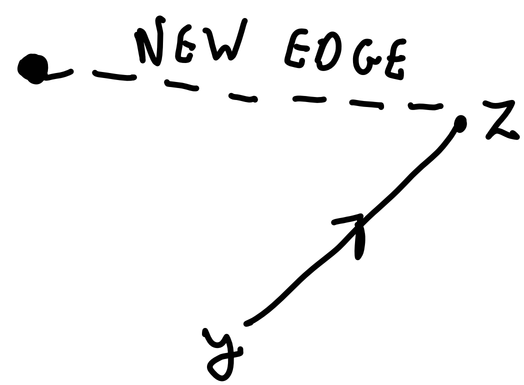
Let

$$Y_i = (x, v) \qquad x > v$$

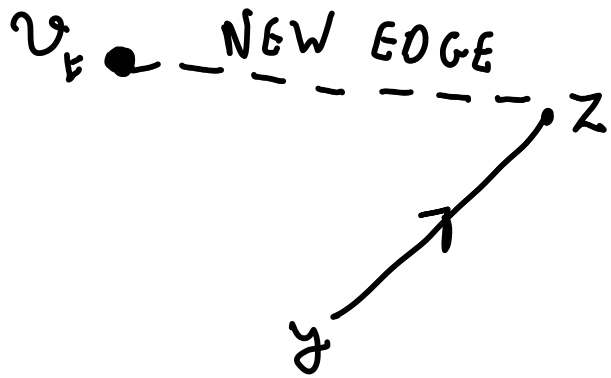
$$\hat{Y}_i = (\hat{x}, \hat{v}) \qquad \hat{x} > \hat{v}$$

$[\ x = \hat{x}$ if $i \bmod m \neq 1\ ]$

Now suppose $j > i$ and $Y_j = (y, z)$. Then

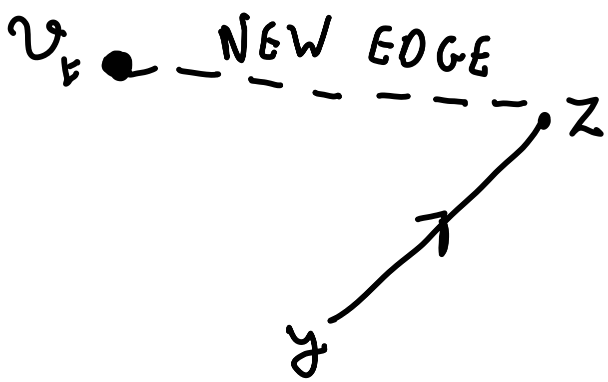


Now suppose $j > i$ and $Y_j = (y, z)$. Then

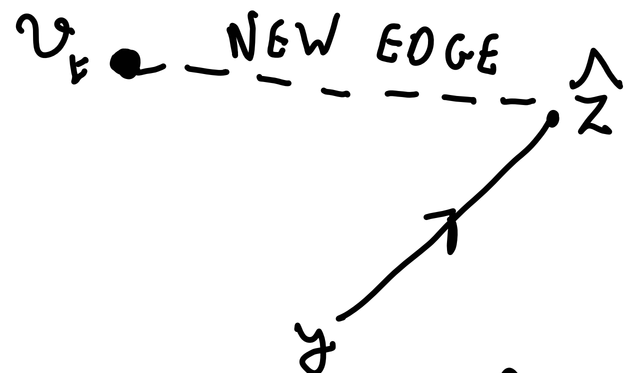


Only $x, \hat{x}, v, \hat{v}$ change degree in transformation.

In $\hat{\ }$ world



or



If (y, z) exists else

$z = v \Rightarrow \hat{z} = \hat{v}$

$z = x \Rightarrow \hat{z} = \hat{x}$

Maximum Degree

Fix $k \leq t$ and let X_ℓ = degree of v_k in Γ_ℓ.

Lemma

$$\Pr\left(X_{mt} \geq A(t/k)^{\frac{1}{2}}(\log t)^2\right) = O\left(t^{-A/2}\right).$$

Proof

$$X_{mk} \leq 2m.$$

If $0 < \lambda < \frac{1}{\log t}$ then

$$E\left(e^{\lambda X_{\ell+1}} \mid X_\ell\right) = e^{\lambda X_\ell}\left(1 - \frac{X_\ell}{2\ell} + \frac{X_\ell}{2\ell} e^{\lambda}\right)$$

$$\leq e^{\lambda X_\ell}\left(1 - \frac{X_\ell}{2\ell} + \frac{X_\ell}{2\ell}\left(1 + \lambda(1+\lambda)\right)\right)$$

$$\leq e^{\lambda\left(1 + \frac{1+\lambda}{2\ell}\right) X_\ell}$$

So if we define a sequence

$\lambda = \lambda_{m_b}, \lambda_{m_b+1}, \dots \lambda_{m_t}$

where

$$\lambda_{j+1} = \left(1 + \frac{1+\lambda_j}{2j}\right)\lambda_j < 1/\log t$$

then

$$E\left(e^{\lambda X_{mt}}\right) \leq E\left(e^{\lambda_{m\ell+1} X_{mt-1}}\right)$$

$$\vdots$$

$$\leq E\left(e^{\lambda_{mt} X_{m\ell}}\right)$$

$$\leq e^{2m/\log t}.$$

$$\lambda_{j+1} \leq \left(1 + \frac{1 + 1/\log t}{2j}\right) \lambda_j$$

implies that

$$\lambda_{mt} \leq \lambda_{ml} \prod_{j=ml}^{mt} \left(1 + \frac{1+1/\log t}{2j}\right)$$

$$\leq \lambda_{ml} \exp\left\{ \sum_{j=ml}^{mt} \frac{1+1/\log t}{2j} \right\}$$

$$\leq 2(t/l)^{1/2} \lambda_{ml} .$$

So argument works for

$$\lambda_{ml} = \frac{(l/t)^{1/2}}{2 \log t} .$$

This gives

$$E\left(\exp\left\{ \underbrace{\frac{(\ell/t)^{1/2}}{2\log t}}_{\lambda} X_{mt} \right\}\right) \leq e^{2m/\log t}$$

Finally,

$$\Pr\left(X_{mt} \geq A\,(t/\ell)^{1/2}(\log t)^2 \right)$$

$$\leq e^{-\lambda A (t/\ell)^{1/2} (\log t)^2} E\left(e^{\lambda X_{mt}}\right)$$

$$\leq t^{-A/2} e^{2m/\log t}.$$

$\square$