Small Trees

Fix $k \geq 3$.

$$m \leq \frac{n^{\frac{k-2}{k-1}}}{\omega} \Rightarrow G_m \text{ contains no tree with } k \text{ vertices.}$$

$$p = \frac{m}{N} \approx \frac{2}{\omega n^{k/(k-1)}}$$

Let

X_k = # of trees with k vertices in $G_{n,p}$

$$E(X_k) = \binom{n}{k} k^{k-2} p^{k-1}$$

$$\leq \left(\frac{ne}{k}\right)^k k^{k-2} \left(\frac{3}{\omega n^{k/(k-1)}}\right)^{k-1}$$

$$< \left(\frac{3e}{\omega}\right)^{k-1}$$

$$\to 0.$$

$$\Pr(G_{n,p} \underbrace{\text{contains tree with } k \text{ vertices}}_{\text{monotone}}) \to 0$$

$$\Downarrow$$

$$\Pr(G_m \text{ contains a tree with } k \text{ vertices}) \to 0.$$

$$m = \omega\, n^{\frac{k-2}{k-1}}, \quad m = o(n)$$

$\Rightarrow$ G_m contains a copy of <u>every</u> tree with k vertices.

$p = \frac{m}{N}$.

Fix some tree T with k vertices.

X_T = # of isolated copies of T in $G_{n,p}$.

$$E(X_T) = \binom{n}{k} \frac{k!}{\text{aut}(T)}{}^{*} p^{k-1} (1-p)^{k(n-k)}$$

$$\simeq^{**} \frac{(2w)^{k-1}}{\text{aut}(T)}$$

$$\to \infty.$$

*aut(H) = no. of automorphisms of H

** = $(1+o(1))$ times .....

Let $\mathcal{T}$ be the set of copies of T in K_n.

$$E(X_T^2) = \sum_{T_1, T_2 \in \mathcal{T}} \Pr\left(T_2 \overset{i}{\subseteq} G_p \mid T_1 \overset{i}{\subseteq} G_p\right) \times \Pr\left(T_1 \overset{i}{\subseteq} G_p\right)$$

$$= E(X_T)\Bigg(1 + \sum_{\substack{T_2 \in \mathcal{T} \\ V(T_2) \cap [k] = \emptyset}} \Pr\left(T_2 \subseteq G_p \mid \hat{T} \subseteq G_p\right)\Bigg)$$

fixed copy of T on $[k]$.

$$\leq E(X_T)\left(1 + (1-p)^{-k} E(X_T)\right).$$

$$\Pr(X_T \neq 0) \geq \frac{E(X_T)^2}{E(X_T)\left(1+(1-p)^{-k}E(X_T)\right)} \longrightarrow 1.$$

$\Pr(G_{n,p}$ contains isolated copy of $T) \to 1$

$\Downarrow$

$\Pr(G_{n,p}$ contains copy of $T) \to 1$

$\Downarrow$

$\Pr(G_m$ contains copy of $T) \to 1.$

Cycles

$m = O(n) \Rightarrow G_m$ is a forest, whp

Suppose $m = n/\omega$

$$p = \frac{m}{N} \leq \frac{3}{\omega n}$$

X = # of cycles in $G_{n,p}$

$$E(X) = \sum_{k=3}^{n} \binom{n}{k} \frac{(k-1)!}{2} p^k$$

$$\leq \sum_{k=3}^{n} \frac{n^k}{2k} \cdot \frac{3^k}{\omega^k n^k}$$

$$= O(\omega^{-3})$$

$$\to 0.$$

$$\Pr(G_{n,p} \text{ is not a forest}) = o(1)$$

$$\Downarrow$$

$$\Pr(G_m \text{ is not a forest}) = o(1).$$

Poisson Convergence.

What happens if

$$m = c\, n^{(k-2)/(k-1)}$$

where $c > 0$ is constant?

Inclusion-Exclusion.

<u>Lemma</u>

Suppose $A_1, A_2, \dots, A_r$ are events in some probability space Ω.

Suppose that $f_1, f_2, \dots, f_s$ are boolean functions of $A_1, A_2, \dots, A_s$

Suppose $\alpha_1, \alpha_2, \dots \alpha_s$ are reals. Then if

$$\sum_{i=1}^{s} \alpha_i \Pr(f_i(A_1, A_2, \dots, A_r)) \geq 0 \qquad (1)$$

whenever $\Pr(A_i) = 0$ or 1 then $(*)$ holds in general.

Write

$$f_i = \bigcup_{S \in T_i} \left(\left(\bigcap_{i \in S} A_i \right) \cap \left(\left(\bigcap_{i \notin S} \bar{A}_i \right) \right) \right)$$

so that

$$Pr(f_i) = \sum_{S \in T_i} Pr\left(\left(\bigcap_{i \in S} A_i \right) \cap \left(\bigcap_{i \notin S} \bar{A}_i \right) \right)$$

and then LHS (1) becomes

$$\sum_{S \subseteq [r]} \beta_S \, Pr\left(\left(\bigcap_{i \in S} A_i \right) \cap \left(\bigcap_{i \notin S} \bar{A}_i \right) \right)$$

for some real β_S.

If (1) holds then $\beta_S \geq 0$, $\forall S$ since we can choose $A_i = \Omega$, $i \in S$, $A_i = \emptyset$ $i \notin S$.

$\square$

For $X \subseteq [r]$ let $A_X = \bigcap_{i \in X} A_i$

$$S_t = \sum_{|X|=t} \Pr(A_X)$$

$\mathcal{E} = \{$none of $A_1, A_2, \dots, A_r$ occur$\}$

Lemma

$$\Pr(\mathcal{E}) - \sum_{t=0}^{r} (-1)^t S_t \quad \begin{matrix} \le 0 & r \text{ even} \\ \ge 0 & r \text{ odd} \end{matrix}$$

We only need to check when

$Pr(A_i) = 1 \quad 1 \leq i \leq l$

$Pr(A_i) = 0 \quad l < i \leq r$

$$Pr(\mathcal{E}) = \begin{cases} 1 & l = 0 \\ 0 & l \neq 0 \end{cases}$$

$$S_t = \binom{l}{t}$$

$l = 0$ trivial.

$\underline{l > 0}$

$$0 - \sum_{t=0}^{r} (-1)^t \binom{l}{t}$$

$$= \begin{cases} 0 & r \geq l \\ (-1)^r \binom{l-1}{r} & r < l \end{cases}$$

$\square$

Back to random graphs

Let $T_1, T_2, \dots T_M$ be the list of copies of some fixed k vertex tree T.

$A_i = \{ T_i$ occurs as a component in $G_m \}$

Suppose $X \subseteq [M]$ with $|X| = t$, t fixed.

$Pr(A_X) = 0$ if $\exists\, i, j \in X$ such that T_i, T_j share a vertex.

Suppose T_i, $i \in X$ are vertex disjoint.

$$\Pr(A_X) = \frac{\binom{\binom{n-kt}{2}}{m-(k-1)t}}{\binom{N}{m}}$$

Numerator = # ways of choosing m edges so that A_X occurs

Now

$$\frac{A^B}{B!} \geq \binom{A}{B} = \frac{A^B}{B!}\left(1-\frac{1}{A}\right)\left(1-\frac{2}{A}\right)\cdots\left(1-\frac{B-1}{A}\right)$$

$$\geq \frac{A^B}{B!}\left(1-\frac{B^2}{2A}\right)$$

So if A, B are functions of n and

$$\frac{B^2}{A} \to \infty \quad \text{as} \quad n \to \infty$$

then

$$\binom{A}{B} = (1+o(1))\frac{A^B}{B!}$$

Consider $\binom{\binom{n-kt}{2}}{m-(k-1)t}$ for $t \le \log n$ say.

$$\binom{n-kt}{2} = N\left(1-\frac{kt}{n}\right)\left(1-\frac{kt}{n-1}\right)$$

$$= N\left(1-O(kt/n)\right)$$

So

$$\frac{m^2}{\binom{n-kt}{2}} \to 0 \text{ and}$$

$$\binom{\binom{n-kt}{2}}{m-(k-1)t} = (1+o(1))\,\frac{\left(N\left(1-O(kt/n)\right)\right)^{m-(k-1)t}}{(m-(k-1)t)!}$$

$$= \frac{(1+o(1))\,N^{m-(k-1)t}\,\left(1-O(mkt/n)\right)}{(m-(k-1)t)!}$$

$$= (1+o(1))\,\frac{N^{m-(k-1)t}}{(m-(k-1)t)!}$$

Similarly

$$\binom{N}{m} = (1+o(1))\,\frac{N^m}{m!}$$

and so

$$\Pr(A_x) = \frac{\binom{\binom{n-kt}{2}}{m-(k-1)t}}{\binom{N}{m}}$$

$$= (1+o(1))\,\frac{m!}{(m-(k-1)t)!}\,N^{-(k-1)t} = (1+o(1))\left(\frac{m}{N}\right)^{(k-1)t}.$$

So

$$S_t \simeq \frac{1}{t!}\binom{n}{k, k, k, \ldots, k}\left(\frac{k!}{\mathrm{aut}(T)}\right)^t \left(\frac{m}{N}\right)^{(k-1)t}$$

$$\simeq \frac{n^{kt}}{t!(k!)^t} \cdot \left(\frac{k!}{\mathrm{aut}(T)}\right)^t \cdot \left(\frac{cn^{(k-2)/(k-1)}}{N}\right)^{(k-1)t}$$

$$\simeq \frac{\lambda^t}{t!}$$

where $\lambda = \dfrac{(2c)^{k-1}}{\mathrm{aut}(T)}$

Fix r large

$$\Pr(\nexists \text{ component copy of } T) =$$

$$\sum_{t=0}^{r} (-1)^t S_t + \theta_r$$

$$= \sum_{t=0}^{r} (-1)^t (1+o(1)) \frac{\lambda^t}{t!} + \theta_r$$

$$= (1+o(1)) \sum_{t=0}^{r} (-1)^t \frac{\lambda^t}{t!} + \theta_r$$

[Here r can be thought of as a large constant while $n \to \infty$.]

$$(1+o(1))\sum_{t=0}^{2r-1}(-1)^t\frac{\lambda^t}{t!} \leq$$

$$\Pr(\nexists \text{ component copy of } T)$$

$$\leq (1+o(1))\sum_{t=0}^{2r}(-1)^t\frac{\lambda^t}{t!}$$

Letting $r \to \infty$

$$\Pr(\nexists \text{ component copy of } T) \Rightarrow e^{-\lambda}$$

If there is a copy of T which is not a component then either

(i) $\exists$ cycle — $Pr((i)) = o(1)$

(ii) T is part of a tree of size $> k$ — $Pr((ii)) = o(1)$.

So

$$Pr(\exists \text{ copy of } T) \geqslant 1 - e^{-\lambda}.$$