Eigenvalues of Random Graphs

Theorem

Suppose $(\ln n)^5 \leq np \leq n - (\ln n)^5$.

Let A denote the adjacency matrix of $G_{n,p}$. Let the eigenvalues of A be $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then whp

(i) $\lambda_1 \approx np$

(ii) $|\lambda_i| \leq \underbrace{2(\log n)^2}\sqrt{np(1-p)}$ $\qquad 2 \leq i \leq n.$

With more work can be replaced by $2+o(1)$.

Main Lemma

Let J be the all 1's matrix and $M = pJ - A$. Then whp

$$\|M\| \leq 2(\log n)^2 \sqrt{np(1-p)}$$

$$\|M\| = \max_{|x|=1} |Mx| = |\lambda_1(M)|$$

We first show that the lemma implies the theorem.

Let $\underline{e}$ denote the all 1's vector.

(a) $$\begin{aligned} |A\underline{e} - np\underline{e}| &= |M\underline{e}| \\ &\leq \|M\| \cdot |\underline{e}| \\ &\leq 2(\log n)^2 n\sqrt{p(1-p)} \end{aligned}$$

(b) Now suppose that $|\xi|=1$ and $\xi \perp \underline{e}$. Then $J\xi = 0$ and

$$|A\xi| = |M\xi| \leq \|M\| \leq 2(\log n)^2 \sqrt{np(1-p)}$$

Now let $|x| = 1$ and let $x = \alpha u + \beta y$ where $u = \frac{1}{\sqrt{n}} \mathbb{1}$ and $y \perp \underline{e}$ and $|y| = 1$. Then

$$|Ax| \leqslant |\alpha|\,|Au| + |\beta|\,|Ay|$$

We have

$$|Au| = \frac{1}{\sqrt{n}} |A\underline{e}| \leqslant \frac{1}{\sqrt{n}} \left(np|\underline{e}| + \|M\| \cdot |\underline{e}| \right)$$

$$\leqslant np + 2(\log n)^2 \sqrt{np(1-p)}$$

$$|Ay| \leqslant 2(\log n)^2 \sqrt{np(1-p)}$$

Thus

$$|Ax| \leqslant |\alpha|\, np + 2\left(|\alpha| + |\beta|\right)(\log n)^2 \sqrt{p(1-p)}$$

$$\leqslant np + 3(\log n)^2 \sqrt{p(1-p)}.$$

This implies that $\lambda_1 \leq (1+o(1))\,np$

But

$$|Au| \geq |(A+M)u| - |Mu|$$

$$= |pJu| - |Mu|$$

$$\geq np - 2(\log n)^2\sqrt{np(1-p)}$$

implying $\lambda_1 \geq (1-o(1))\,np$.

Now

$$\lambda_2 = \min_{\eta} \max_{0 \neq \xi \perp \eta} \frac{|A\xi|}{|\xi|}$$

$$\leq \max_{0 \neq \xi \perp e} \frac{|A\xi|}{|\xi|}$$

$$\leq 2(\log n)^2 \sqrt{np(1-p)}$$

$$\lambda_n = \min_{|\xi|=1} \xi^T A \xi \geq \min_{|\xi|=1} \xi^T A \xi - p \overbrace{\xi^T J \xi}^{\geq 0}$$

$$= \min_{|\xi|=1} -\xi^T M \xi \geq -\|M\| \geq -2(\log n)^2 \sqrt{np(1-p)}$$

Proof of Main Lemma

Putting $\hat{M} = M - pI_n$ (zero on diagonal)

we see that

$$\|M\| \leq \|\hat{M}\| + \|pI_n\| = \|\hat{M}\| + p$$

and so we bound $\|\hat{M}\|$.

Letting m_{ij} denote (i,j) entry of $\hat{M}$ we have

(i) $E(m_{ij}) = 0$

(ii) $Var(m_{ij}) \leq p(1-p) \leftarrow \sigma^2$.

(iii) $m_{ij}, m_{i'j'}$ are independent, unless $(i',j') = (j,i)$.

Now let $k \geq 2$ be an even integer.

$$\mathrm{Trace}(\hat{M}^k) = \sum_{i=1}^{n} \lambda_i(\hat{M})^k$$

$$\geq \max\{\lambda_1(\hat{M})^k, \lambda_n(\hat{M})^k\}$$

$$= \|\hat{M}\|^k.$$

We estimate

$$\|\hat{M}\| \leq \mathrm{Trace}(\hat{M}^k)^{1/k}$$

where

$$k = (\log n)^2.$$

$$E\left(\mathrm{Trace}(\hat{M}^k)\right) = \sum_{i_0=1}^{n} \sum_{i_1=1}^{n} \cdots \sum_{i_{k-1}=1}^{n} E\left(m_{i_0 i_1} m_{i_1 i_2} \cdots m_{i_{k-2} i_{k-1}} m_{i_{k-1} i_0}\right)$$

So

$$\|\hat{M}\|^k \leq \sum_{\rho=2}^{k+1} E_{n,k,\rho}$$

where

$$E_{n,k,\rho} = \sum_{\substack{i_0=1 \\ |\{i_0, i_1, \ldots, i_{k-1}\}| = \rho}}^{n} \sum_{i_1=1}^{n} \cdots \sum_{i_{k-1}=1}^{n} \left| E\left(\prod_{j=0}^{k-1} m_{i_j i_{j+1}}\right)\right|$$

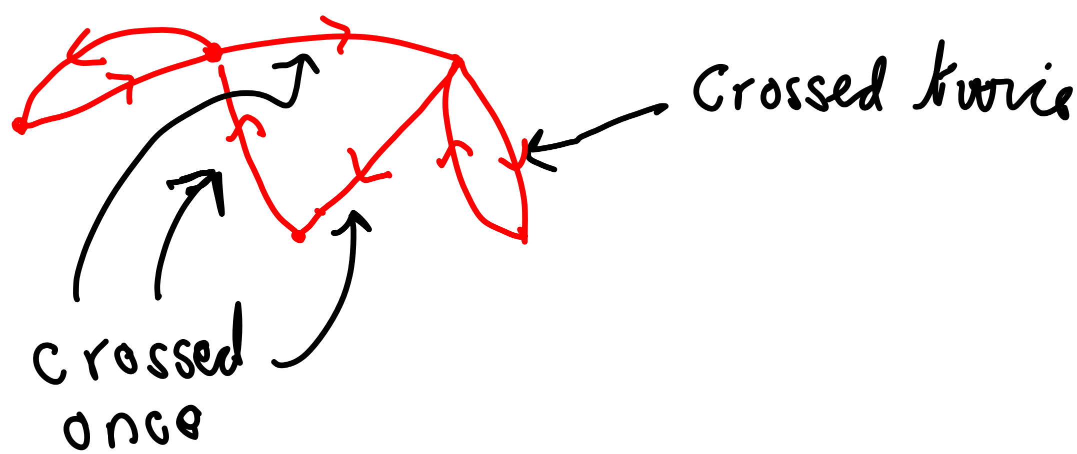
Note that $m_{i,i}=0$ implies $E_{n,k,1}=0$.

Each sequence $\underline{i} = i_0 i_1 \ldots, i_{k-1}, i_0$ corresponds to a walk on $W(\underline{i})$ on K_n, with n loops added.

Note that

$$E\left(\prod_{j=0}^{k-1} m_{i_j i_{j+1}}\right) = 0$$

if the walk $W(\underline{i})$ contains an edge that is crossed exactly once



On the other hand, $|m_{i_j}| \leq 1$ and so

$$\left| E\left(\prod_{j=0}^{k-1} m_{i_j i_{j+1}} \right) \right| \leq \sigma^{2(\rho-1)}$$

if each edge of $W(\underline{i})$ is crossed at least twice and if $|\{i_0, i_1, \ldots, i_{k-1}\}| = \rho$.

Let $R_{k,\rho}$ denote the number of (k,ρ)-walks.

We use the following trivial estimates:

(i) $\rho > \frac{k}{2}+1$ implies $R_{k,\rho} = 0$

(ii) $\rho \leq \frac{k}{2}+1$ implies

$$R_{k,\rho} \leq n^{\rho} \, k^{k}$$

choose the ρ distinct vertices

number of walks of length k

We have

$$\|\hat{M}\|^k \le \sum_{\rho=2}^{\frac{1}{2}k+1} R_{k,\rho}\, \sigma^{2(\rho-1)}$$

$$\le \sum_{\rho=2}^{\frac{1}{2}k+1} n^{\rho} k^{k} \sigma^{2(\rho-1)}$$

$$\le 2 n^{\frac{1}{2}k+1} k^{k} \sigma^{k}.$$

Thus

$$E(\|\hat{M}\|^k) \le 2n^{\frac{1}{2}k+1} k^k \sigma^k$$

Then

$$\Pr(\|\hat{M}\| \ge 2k\sigma n^{\frac{1}{2}})$$

$$= \Pr(\|\hat{M}\|^k \ge (2k\sigma n^{\frac{1}{2}})^k)$$

$$\le \frac{E(\|M\|^k)}{(2k\sigma n^{\frac{1}{2}})^k}$$

$$\leq \frac{2n^{\frac{1}{2}k+1}\, k^{k}\, \sigma^{k}}{(2k\sigma n^{1/2})^{k}}$$

$$= \left(\frac{(2n)^{1/k}}{2}\right)^{k}$$

$$= \left(\frac{1}{2} + o(1)\right)^{k}$$

$$= o(1).$$