Differential Equations Method

Consider the following simple process:
We start with n isolated vertices
$1, 2, \dots, n$.

At a general step, we choose a (still) isolated vertex v and add an edge to a randomly chosen w.

Question: how long before there are no isolated vertices?

Let

$$X(t) = \#\text{ isolated vertices after } t \text{ steps.}$$

$$X(0) = n$$

$$E\big(X(t+1) - X(t) \mid X(t)\big) = -1 - \frac{X(t)}{n-1}. \qquad (*)$$

Now put $t = \tau n$, $0 \leq \tau \leq 1$

and $n x(\tau) = X(t)$.

(*) on p2 suggest that

$$x'(\tau) = -1 - x(\tau)$$

given

$$x(\tau) = 2e^{-\tau} - 1.$$

In which case we would expect that the process ends when $t \simeq n \ln 2$.

We now consider the following greedy algorithm for finding an independent set in a graph.

GREEDY

begin

$I \leftarrow \emptyset$; $A \leftarrow V$;

While $A \neq \emptyset$ **do**

Choose $v \in A$; [Random Choice]

$I \leftarrow I \cup \{v\}$; $A \leftarrow A \setminus (\{v\} \cup N(v))$

od

Output I

end

Greedy produces an independent set.
We begin by studying the likely size of the output, if G is a random r-regular graph.

We use the configuration model of r-regular graphs i.e. $W = W_1 \cup W_2 \cup \cdots \cup W_n$ where $W_i = [(i-1)r+1, ir]$

We will expose the random pairing of W as the algorithm progresses i.e. not before.

If vertex i is placed in the independent set I, then and only then, do we expose the pairs involving W_i.

Let the degree of a vertex j at a general step of the algorithm be the number of exposed pairs involving W_j

Thus a general step of GREEDY involves

(i) Choose a vertex i of degree zero.

(ii) Expose the pairs involving W_i.

Let $t = |I|$ be the number of steps taken so far and let P_t refer to the current set of exposed pairs.

Let $X(t)$ be the number of vertices of degree zero.

The number of vertices in the set chosen by GREEDY is t_0, where $X(t_0) = 0$.

$$E\left(X(t+1)-X(t) \mid P_t\right) =$$

$$-1 - \frac{X(t)\,r}{n-2t} + O\left(\frac{1}{\alpha n}\right) \qquad (*)$$

$v \in I$

assuming $t \le \left(\frac{1}{2}-\alpha\right)n$

We expose r pairs associated with v. For first pair there are still $r(X(t)-1)$ points associated with vertices of degree zero, (excluding v). There are $rn-2rt$ points unpaired altogether. So the probability of pairing with vertex of degree zero is $\frac{r(X(t)-1)}{rn-2rt-1} = \frac{X(t)}{n-2t} + O\left(\frac{1}{n}\right)$.

Repeat r times to get $(*)$.

Putting $t = \tau n$ and $X(t) = n x(\tau)$, this suggests that we solve

$$x'(\tau) = -1 - \frac{r\, x(\tau)}{1 - 2\tau}$$

$$x(0) = 1.$$

Solution: $x(\tau) = \dfrac{(r-1)(1-2\tau)^{r/2} - (1-2\tau)}{r-2}$

The smallest positive solution to $x(\tau) = 0$ is

$$\tau_0 = \frac{1}{2}\left(1 - \left(\frac{1}{r-1}\right)^{2/(r-2)}\right)$$

and then number of vertices in independent set chosen by GREEDY is whp, $\approx \tau_0 n$.

For the following:

$q_0, q_1, \ldots\ldots q_t, \ldots q_n \in S$ is a random process.

$H_t = (q_0, q_1, \ldots, q_t)$ is the history to time t.

$X(0), X(1), \ldots X(t), \ldots$ are random variables where

$X(t) = X_t(H_t)$.

$D \subseteq \mathbb{R}^2$ is open and connected and

$$\left(0, \frac{X_0(q_0)}{n}\right) \in S$$

[We can assume

[q_0 is fixed

We further assume

(i) $|X(t)| \leq C_0 n,$ $\forall\, t < T_D$ where C_0 is constant.

(ii) $|X(t+1) - X(t)| \leq \beta = \beta(n) \geq 1, \forall\, t < T_D$

(iii) $\left| E\left(X(t+1) - X(t) \,\middle|\, H_t \right) - f\left(t/n, X(t)/n\right) \right| \leq \lambda_0,$

$\forall\, t < T_D$

(iv) $f(t, x)$ is continuous and satisfies a Lipschitz condition on $D \cap \{(t, x) : t \geq 0\}$

i.e. $|f(\underline{x}) - f(\underline{x}')| \leq L \, \|\underline{x} - \underline{x}'\|_\infty .$

Example 1

$H_t = (i_1, i_2), (i_3, i_4), \ldots (i_{2t-1}, i_{2t}) \quad 1 \le i_k \le n$

$X_t(H_t) = n - |\{i_1, i_2, \ldots, i_{2t}\}|$

$f(t, x) = -1 - x$

$D = (-1, 1)^2$

$C_0 = 1$

$\lambda_0 = \frac{1}{n-1}$

$L = 1$

Example 2

$W = [rn]$

$H_t = (i_1, i_2), (i_3, i_4), \ldots (i_{2t-1}, i_{2t}) \quad 1 \le i_k \le rn$

$X_t(H_t) = n - |\{a : \exists s \text{ s.t. } i_s \in W_a\}|$

$f(t, x) = -1 - \frac{rx}{1-2t}$

$C_0 = 1$

$\lambda_0 = \frac{1}{2n}$

$L = r/2\alpha$

$D = (-1, \frac{1}{2} - \alpha) \times (0, 1)$

Theorem

Suppose $\lambda > \lambda_0$ and C is sufficiently large and

$\sigma = \inf\{\tau : (\tau, z(\tau)) \notin D_0 = \{(t,z) \in D : \ell^\infty$ distance of (t,z) to boundary of $D \geq C\lambda\}\}$

Here $z(t)$, $0 \leq \tau \leq \sigma$ be the unique solution to

$$\dot{z}(\tau) = f(\tau, x) \qquad (*)$$

$$z(0) = \frac{X_0(q_0)}{n}$$

With probability $1 - O\left(\frac{\beta}{\lambda}\exp\left(-\frac{n\lambda^3}{\beta^3}\right)\right)$

$$X(t) = n\, z(t/n) + O(\lambda n)$$

uniformly in $0 \leq t \leq \sigma n$.

Proof

Let $\quad w = \left\lceil \frac{n\lambda}{\beta} \right\rceil$.

We can assume that $\lambda/\beta \geq n^{-1/3}$ else there is nothing to prove.

We study the concentration of $X(t+w) - X(t)$, so assume that $(t/n, X(t/n)) \in D_0$.

For $0 \leq k \leq w$ we have

Note that $\left| \frac{X(t+k)}{n} - \frac{X(t)}{n} \right| \leq \frac{k\beta}{n} \leq 2\lambda$

So $\left\| \underbrace{\left(\frac{t+k}{n}, \frac{X(t+k)}{n} \right)} - \left(\frac{t}{n}, \frac{X(t)}{n} \right) \right\|_\infty \leq 2\lambda$

and so the first of these is in D, assuming $C \geq 2$.

$$E\left(X(t+k+1)-X(t+k) \mid H_{t+k}\right) =$$

$$f\left(\frac{t+k}{n}, \frac{X(t+k)}{n}\right) + \theta_k =$$

$|\theta_k| \le \lambda$

$$f\left(\frac{t}{n}, \frac{X(t)}{n}\right) + \psi_k + \theta_k =$$

$|\psi_k| \le \frac{L\beta k}{n}$

$$f\left(\frac{t}{n}, \frac{X(t)}{n}\right) + \rho$$

where $|\rho| \le 2L\lambda$.

Now, given H_t, let

$$Z_k = X(t+k) - X(t) - k f\left(t/n, \frac{X(t)}{n}\right) - 2kL\lambda.$$

Then

$$E(Z_k - Z_{k-1} \mid Z_0, \ldots Z_{k-1}) \leq 0$$

i.e. $Z_0, Z_1, \ldots, Z_\omega$ is a supermartingale.

Also

$$|Z_k - Z_{k-1}| \leq \beta + \left|f\left(t/n, \frac{X(t)}{n}\right)\right| + 2L\lambda$$

$$\leq K_0\beta$$

where $K_0 = O(1)$.

O(1) by continuity and boundedness of S.

So, conditional on H_t,

$$\Pr\left(X(t+w) - X(t) - w f(t/n, X(t/n)) \geq 2Lw\lambda + K_0 \beta \sqrt{2\alpha w} \right)$$
$$\leq e^{-\alpha}.$$

Similarly,

$$\Pr\left(X(t+w) - X(t) - w f(t/n, X(t/n)) \leq -2Lw\lambda - K_0 \beta \sqrt{2\alpha w} \right)$$
$$\leq e^{-\alpha}.$$

Here we produce a supermartingale or equivalently consider $-X(t)$.

Thus

$$\Pr\left(\left| X(t+w) - X(t) - w f(t/n, X(t/n)) \right| \geq \overbrace{2Lw\lambda + K_0 \beta \sqrt{2\alpha w}}^{err} \right) \leq 2e^{-\alpha}.$$

We will choose

$$\alpha = \frac{n\lambda^3}{\beta^3}$$

so that $w\lambda$ and $\beta\sqrt{2\alpha w}$ are both $\Theta(n\lambda^2/\beta)$ giving

$$err \leq K_1 \frac{n\lambda^3}{\beta}$$

Now let $k_i = i\,w$ for $i = 0, 1, \ldots, i_0 = \lfloor \sigma n / w \rfloor$.

We will show by induction that

$$\Pr\left(\exists j \le i : \left| X(k_j) - z(k_j/n)\, n \right| \ge B_j\right) \le 2 i e^{-\alpha}$$

where

$$B_j = B\left(\left(1 + \frac{Lw}{n}\right)^j - 1\right) \frac{n\lambda^2}{\beta}$$

and where B is another constant.

The induction begins with $z(0) = \dfrac{X(0)}{n}$.

Note that $B_{i_0} = O\left(\dfrac{n\lambda^3}{\beta}\right) = O(\lambda n)$.

Now write

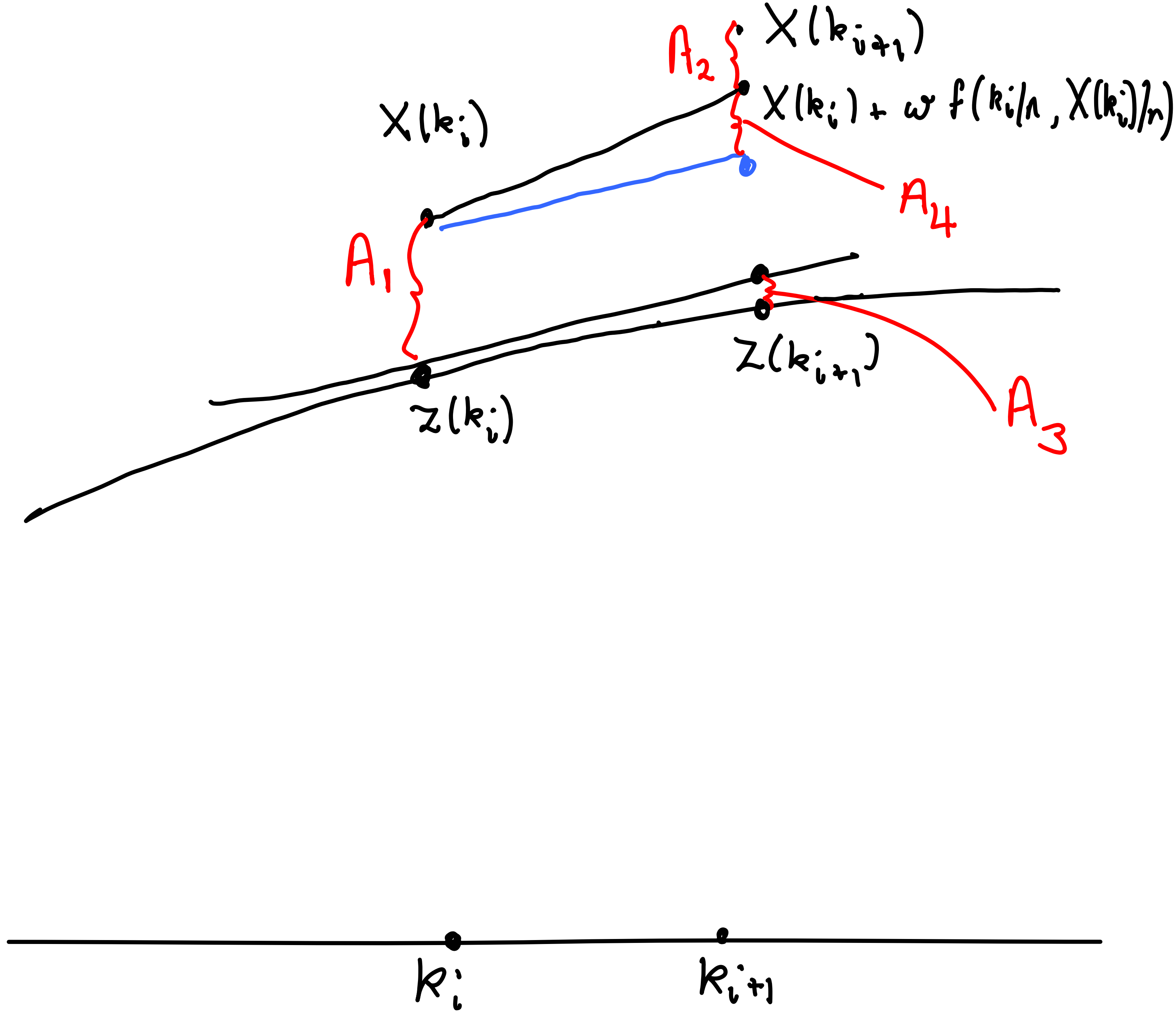
$$|X(k_{i+1}) - z(k_{i+1}/n)\,n| = |A_1 + A_2 + A_3 + A_4|$$

$$A_1 = X(k_i) - z(k_i/n)\,n$$

$$A_2 = X(k_{i+1}) - X(k_i) - w f(k_i/n, X(k_i)/n)$$

$$A_3 = w z'(k_i/n) + z(k_i/n)\,n - z(k_{i+1}/n)\,n$$

$$A_4 = w f(k_i/n, X(k_i)/n) - w z'(k_i/n)$$



$X(k_{i+1})$
A_2
$X(k_i)$
$X(k_i) + \omega f(k_i/n, X(k_i)/n)$
A_4
A_1
$Z(k_{i+1})$
A_3
$Z(k_i)$
k_i
k_{i+1}

$$A_1 = X(k_i) - Z(k_i/n)\, n$$

The induction gives

$$|A_1| \leq B_i .$$

$$A_2 = X(k_{i+1}) - X(k_i) - w f(k_i/n, X(k_i/n))$$

$$|A_2| \le K_1 \frac{n\lambda^3}{\beta}$$

with probability $1 - 2e^{-\alpha}$.

$$A_3 = w z'(k_i/n) + z(k_i/n)\, n - z(k_{i+1}/n)\, n$$

$$|A_3| \le L \frac{w^2}{n^2} \cdot n = L \frac{w^2}{n} \le 2Ln\frac{\lambda^3}{\beta^2}$$

$$A_4 = \omega f(k_i/n, X(k_i)/n) - \omega Z'(k_i/n)$$

$$|A_4| \le \frac{\omega L A_1}{n} \le \frac{\omega L}{n} B_i .$$

Thus, for some $B > 0$,

$$B_{i+1} \le |A_1| + |A_2| + |A_3| + |A_4|$$

$$\le \left(1 + \frac{\omega L}{n}\right) B_i + B n^{\lambda^3/\beta} .$$

Finally consider $k_i \le t < k_{i+1}$.

From "time" k_i to t, the change in X and nZ is at most $\omega\beta = O(n\lambda)$.

$\square$

The above proof generalises easily to the case where

(i) $X(t)$ is replaced by $X_1(t), X_2(t), \dots X_a(t)$ where $a = O(1)$.

(ii) Condition (iii) on P11 holds with probability $1-\gamma$.
This adds $O(n\gamma)$ to the error probability.
We simply condition on (iii) always holding.