Random Graphs with a Fixed Degree Sequence.

Let $\underline{d} = (d_1, d_2, \ldots, d_n)$

where $d_1 + d_2 + \cdots + d_n = 2M$ is even.

Let $\mathcal{G}_{n,\underline{d}} = \{$simple graphs with vertex set $[n]$ such that degree $d(i) = d_i,\ i \in [n]\}$

$G_{n,\underline{d}}$ is chosen randomly from $\mathcal{G}_{n,\underline{d}}$.

We assume that $d_1, d_2, \ldots, d_n \geq 1$ and that $\sum d_i(d_i - 1) = \Omega(n)$.

Configuration model

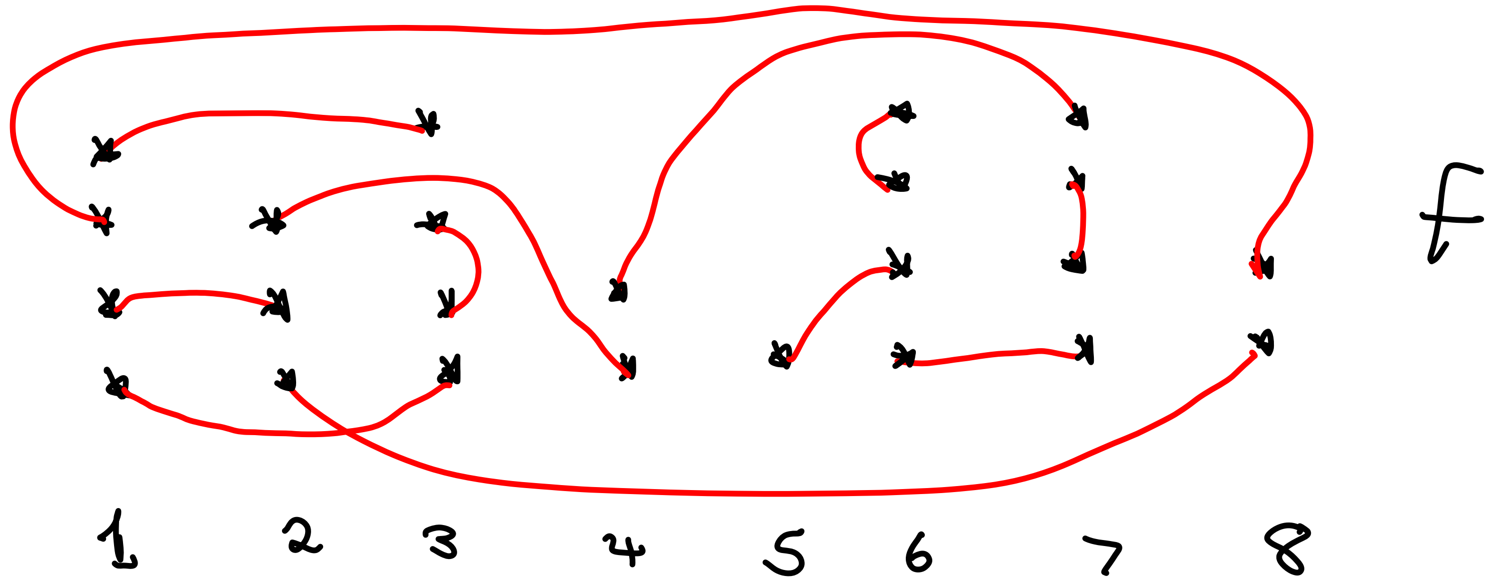
Let $W_1, W_2, \ldots, W_n$ be a partition of W, where $|W_i| = d_i$ for $1 \le i \le n$.

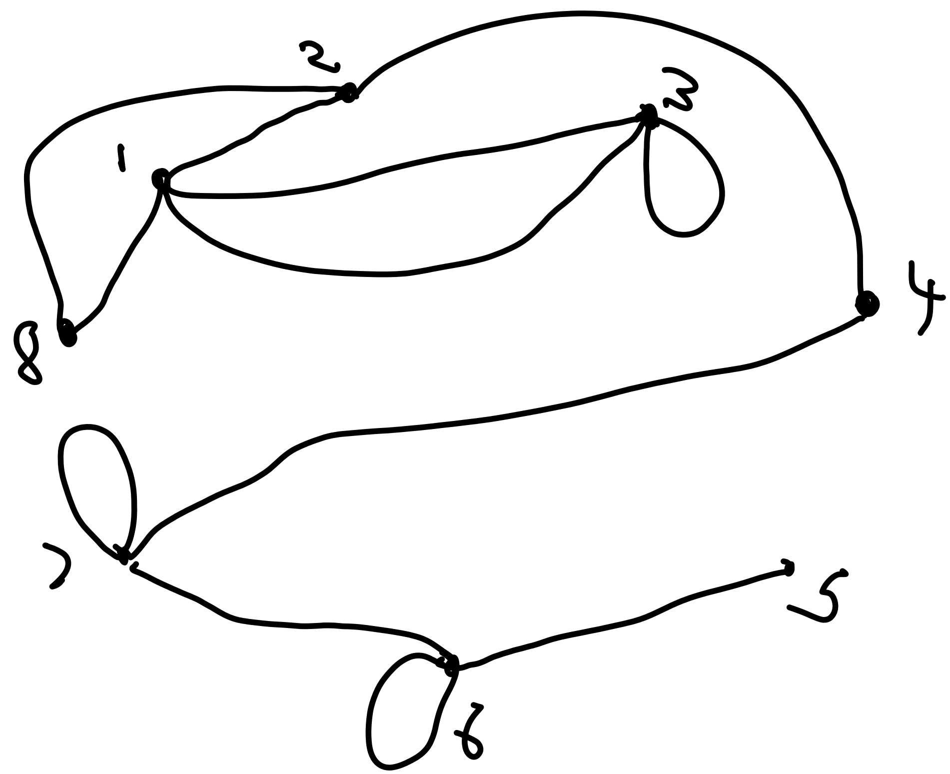
For $x \in W$ define $\phi(x)$ by $x \in W_{\phi(x)}$.

Let F be a partition of W into m pairs (a configuration).

Given F we define the (multi) graph $\gamma(F)$

$$\gamma(F) = \left([n], \{(\phi(x), \phi(y)) : (x, y) \in F\}\right).$$


1
2
3
4
5
6
7
8
F


2
1
3
4
8
7
5
6

Lemma

If $G \in \mathcal{G}_{n,\underline{d}}$, then

$$|\gamma^{-1}(G)| = \prod_{i=1}^{n} d_i!$$

Proof

Arrange the edges of G in lexicographic order. Now go through the sequence of $2m$ symbols, replacing each i by a new member of W_i. We get all F for which $\gamma(F) = G$.

$\square$

Corollary

If F is chosen uniformly at random from Ω (the set of configurations) and $G_1, G_2 \in \mathcal{G}_{n,d}$ then

$$\Pr(\gamma(F) = G_1) = \Pr(\gamma(F) = G_2).$$

So we can choose a random F and accept $\gamma(F)$ iff there are no loops or multiple edges.

The next question is: What is

$$\Pr(\,\gamma(F) \text{ is simple}\,] \qquad \text{(i)}$$

Since

$$|\Omega| = \frac{(2m)!}{m!\,2^m} \qquad \text{(ii)}$$

Take d_i "distinct" copies of i for $i = 1, 2, \dots, n$ and take a permutation of these $2m$ symbols. Read off F, pair by pair.

Each distinct F arises in $m!\,2^m$ ways.

(i) & (ii) will tell us how big is $\mathcal{G}_{n,d}$.

Alternative Construction of F

Begin

$U = W$; $F = \emptyset$;

For $i = 1, 2, \dots, m$ **do**

Begin

Choose x arbitrarily from U;

Choose y randomly from $U \setminus \{x\}$;

$F := F \cup \{(x,y)\}$;

$U := U \setminus \{x, y\}$

End

Each F arises with probability $\dfrac{1}{(2m-1)(2m-3)\cdots 1}$

$= |\Omega|^{-1}$.

Let $\Delta = \max\{d_1, d_2, \ldots, d_n\}$.

Notation: $F \overset{ran}{\in} \Omega$ $\equiv$ F is chosen uniformly from Ω.

Lemma

← not best possible

Assume that $\Delta \leq n^{1/6}$ and $F \overset{ran}{\in} \Omega$. Then whp

(a) $\gamma(F)$ has no double loops

(b) $\gamma(F)$ has $\leq \Delta \log n$ loops

(c) $\gamma(F)$ has no triple edges.

(d) $\gamma(F)$ has no adjacent double edges.

(e) $\gamma(F)$ has $\leq \Delta^2 \log n$ double edges.

Proof

(a)

Pr(F contains a double loop)

$$\leq \sum_{i=1}^{n} \binom{d_i}{4} \cdot 3 \cdot \left(\frac{1}{2m-3}\right)^2$$

$$\leq n \Delta^4 m^{-2}$$

$$= O(1),$$

(b)

Let $k_1 = \Delta \log n$.

$$\Pr(F \text{ has } \geq k_1 \text{ loops}) \leq O(1) + \sum_{\substack{x_1 + \cdots + x_n = k_1 \\ x_i = 0,1}} \prod_{i=1}^{n} \binom{d_i}{2}^{x_i} \left(\frac{1}{2m - 2k_1}\right)^{k_1}$$

$$\leq O(1) + \left(\frac{\Delta}{2m}\right)^{k_1} \sum_{x_1 + x_2 + \cdots + x_n = k_1} \prod_{i=1}^{n} d_i^{x_i}$$

$$\leq O(1) + \left(\frac{\Delta}{2m}\right)^{k_1} \frac{(d_1 + \cdots + d_n)^{k_1}}{k_1!} \leq O(1) + \left(\frac{\Delta e}{k_1}\right)^{k_1} = O(1).$$

(c)

$\Pr(F$ contains a triple edge$)$

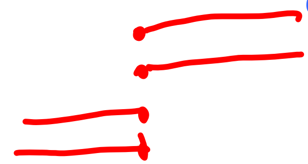
$$\leq \sum_{1 \leq i < j \leq n} \binom{d_i}{3}\binom{d_j}{3} \cdot 6 \cdot \left(\frac{1}{2m-5}\right)^3$$

$$\leq \Delta^4 \left(\sum d_i\right)^2 n^{-3}$$

$= o(1).$

(d)

$$\Pr(F \text{ contains 2 adjacent double edges}) \leq \sum_{i=1}^{n} \binom{d_i}{2}^2 \left(\frac{\Delta}{2m-8}\right)^2$$



a, b $\Pr \leq \frac{\Delta}{2m-8}$ that a, b are in same W_i.

$$\leq \frac{\Delta^3}{(2m-8)^2} \sum_{i=1}^{n} d_i$$

$$= O(1).$$

(e) Let $k_2 = \Delta^2 \log n$.

$$\Pr(F \text{ has} \geq k_2 \text{ double edges})$$

$$o(1) + \sum_{\substack{x_1 + \cdots + x_n = k_2 \\ x_i = 0,1}} \prod_{i=1}^{n} \left[\binom{d_i}{2} \frac{\Delta}{2m - 4k_2} \right]^{x_i}$$

$$o(1) + \left(\frac{\Delta^2}{m}\right)^{k_2} \sum_{\substack{x_1 + \cdots + x_n = k_2 \\ x_i = 0,1}} \prod_{i=1}^{n} d_i^{x_i}$$

$$\leq o(1) + \left(\frac{\Delta^3}{m}\right)^{k_2} \cdot \frac{(2m)^{k_2}}{k_2!}$$

$$= o(1).$$

$\square$

Switchings

Let now

$\Omega_{i,j} = \{ F \in \Omega :$ F has i loops, j double edges and no double loops or triple edges and no vertex incident with 2 double edges $\}$

Lemma (Switchings) Let $M_1 = 2m$ and $M_2 = \sum_i d_i(d_i - 1)$

For $i \le k_1$ and $j \le k_2$

$$\frac{|\Omega_{i-1,j}|}{|\Omega_{i,j}|} = \frac{2iM_1}{M_2}\left(1 + \tilde{O}\left(\frac{\Delta^3}{n}\right)\right)$$

$$\frac{|\Omega_{0,j-1}|}{|\Omega_{0,j}|} = \frac{4jM_1^2}{M_2^2}\left(1 + \tilde{O}\left(\frac{\Delta^3}{n}\right)\right).$$

Corollary

$$\frac{|\Omega_{0,0}|}{|\Omega|} = (1+o(1))\, e^{-\lambda(\lambda+1)}$$

where $\lambda = \dfrac{M_2}{2M_1}$.

Thus

$$|G_{n,\underline{d}}| \approx e^{-\lambda(\lambda+1)} \cdot \frac{1}{\prod_{i=1}^{n} d_i!} \cdot \frac{(2m)!}{m!\, 2^m}.$$

Proof

It follows from the switching lemma that $i \le k_1$ and $j \le k_2$ implies

$$\frac{|\Omega_{i,j}|}{|\Omega_{0,0}|} = (1+o(1)) \frac{\lambda^{i+2j}}{i!\ j!}$$

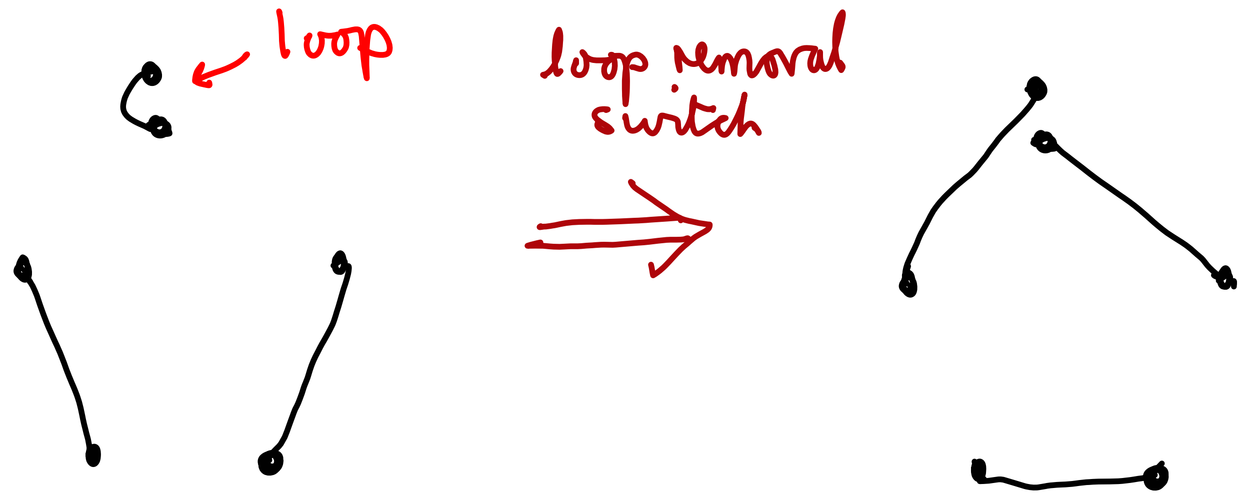
Therefore

$$(1-o(1))|\Omega| = (1+o(1))|\Omega_{0,0}| \sum_{i=0}^{\lambda_1} \sum_{j=0}^{\lambda_2} \frac{\lambda^{i+2j}}{i!\ j!}$$

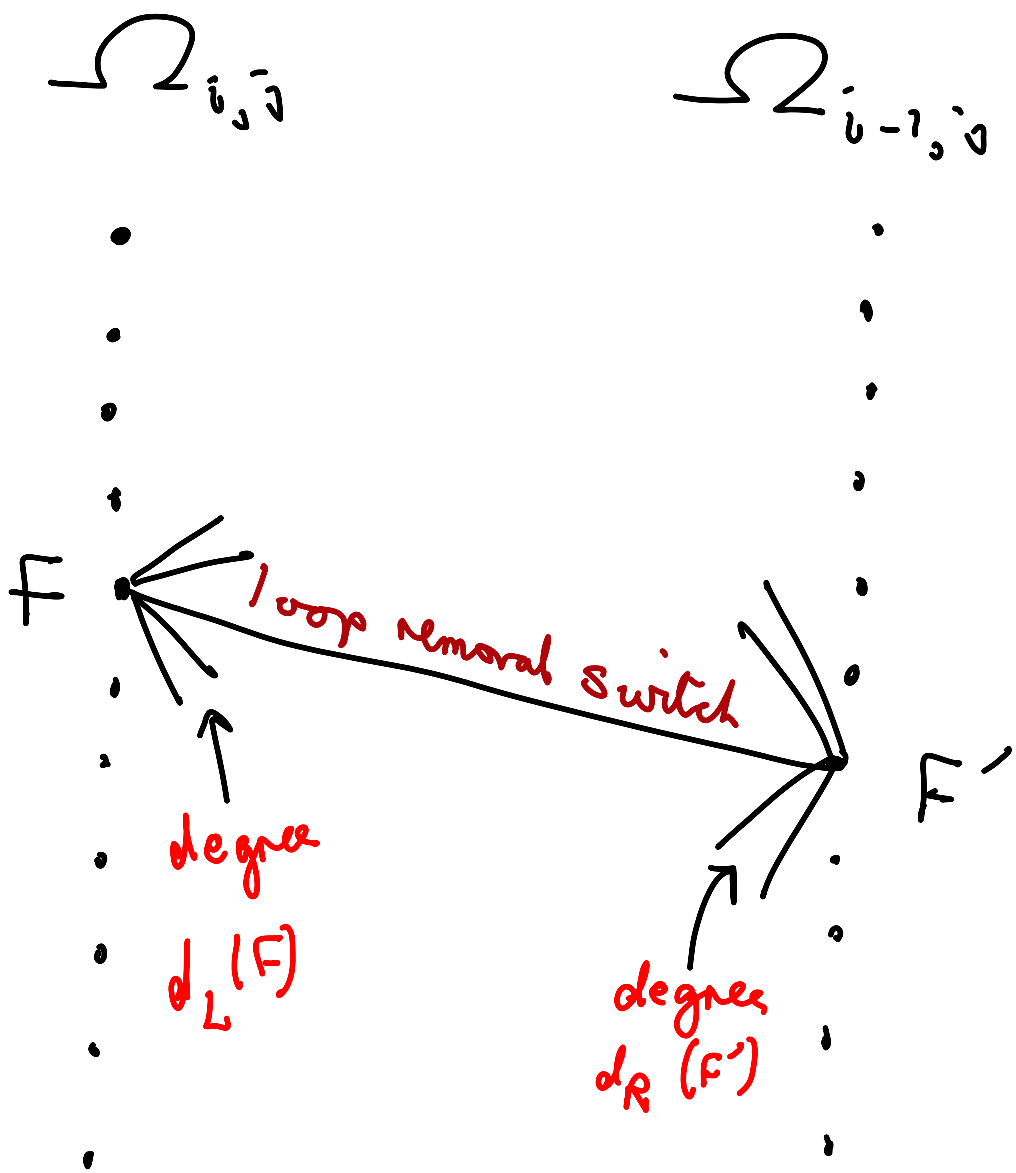
$$= (1+o(1))|\Omega_{0,0}| e^{\lambda(\lambda+1)}.$$

$\square$

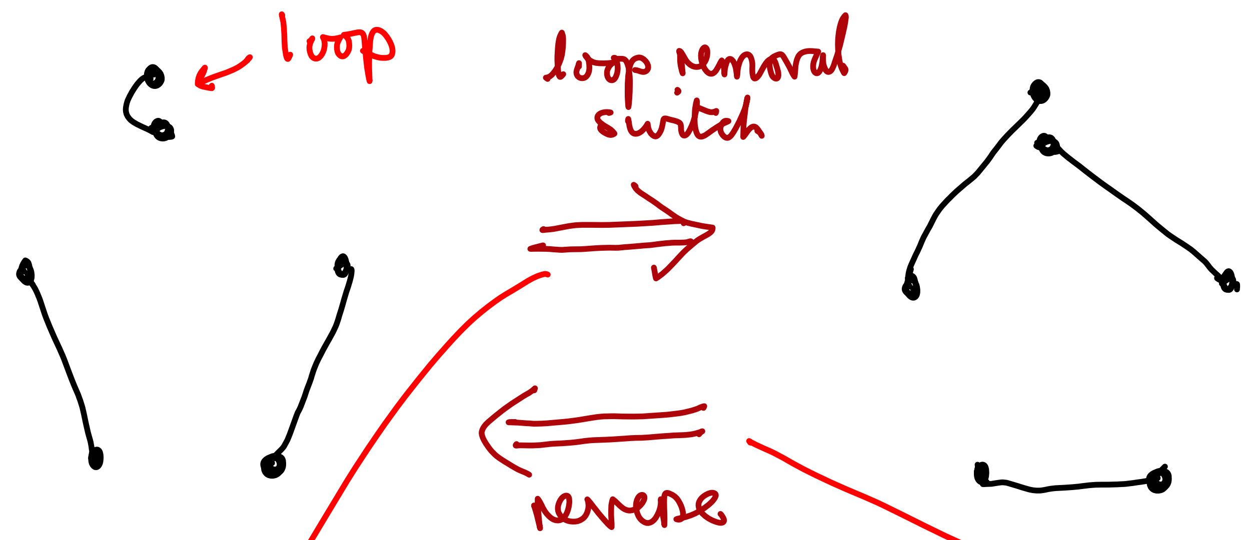
Proof of switching lemma



In general this operation takes a member F of $\Omega_{i,j}$ to a member F' of $\Omega_{i-1,j}$,
unless it creates new loops or multiple edges.



$\Omega_{i,j}$
$\Omega_{i-1,j}$
F
F'
loop removal switch
degree
$d_L(F)$
degree
$d_R(F')$



\# choices : $\leq i \times M_1^2$

$\geq i \times M_1^2 - \tilde{O}(i M_1 \Delta^2)$

\# choices $\leq M_1 M_2 / 2$

$\geq M_1 M_2 / 2 - \tilde{O}((M_1 + M_2)\Delta^3)$

Now

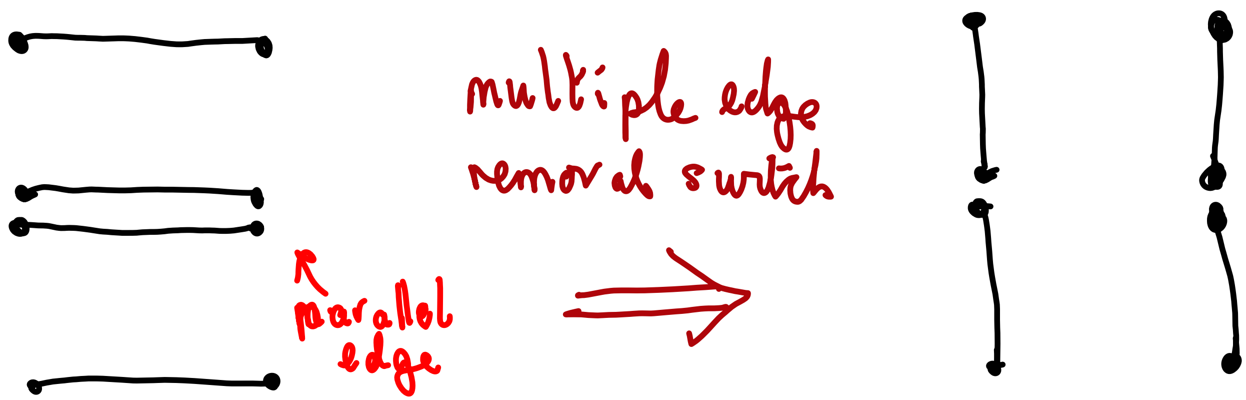
$$\sum_{F\in\Omega_{i,j}} d_L(F) = \sum_{F'\in\Omega_{i-1,j}} d_R(F')$$

$$\le i M_1^2 |\Omega_{i,j}| \qquad\qquad \le \tfrac{1}{2} M_1 M_2 |\Omega_{i-1,j}|$$

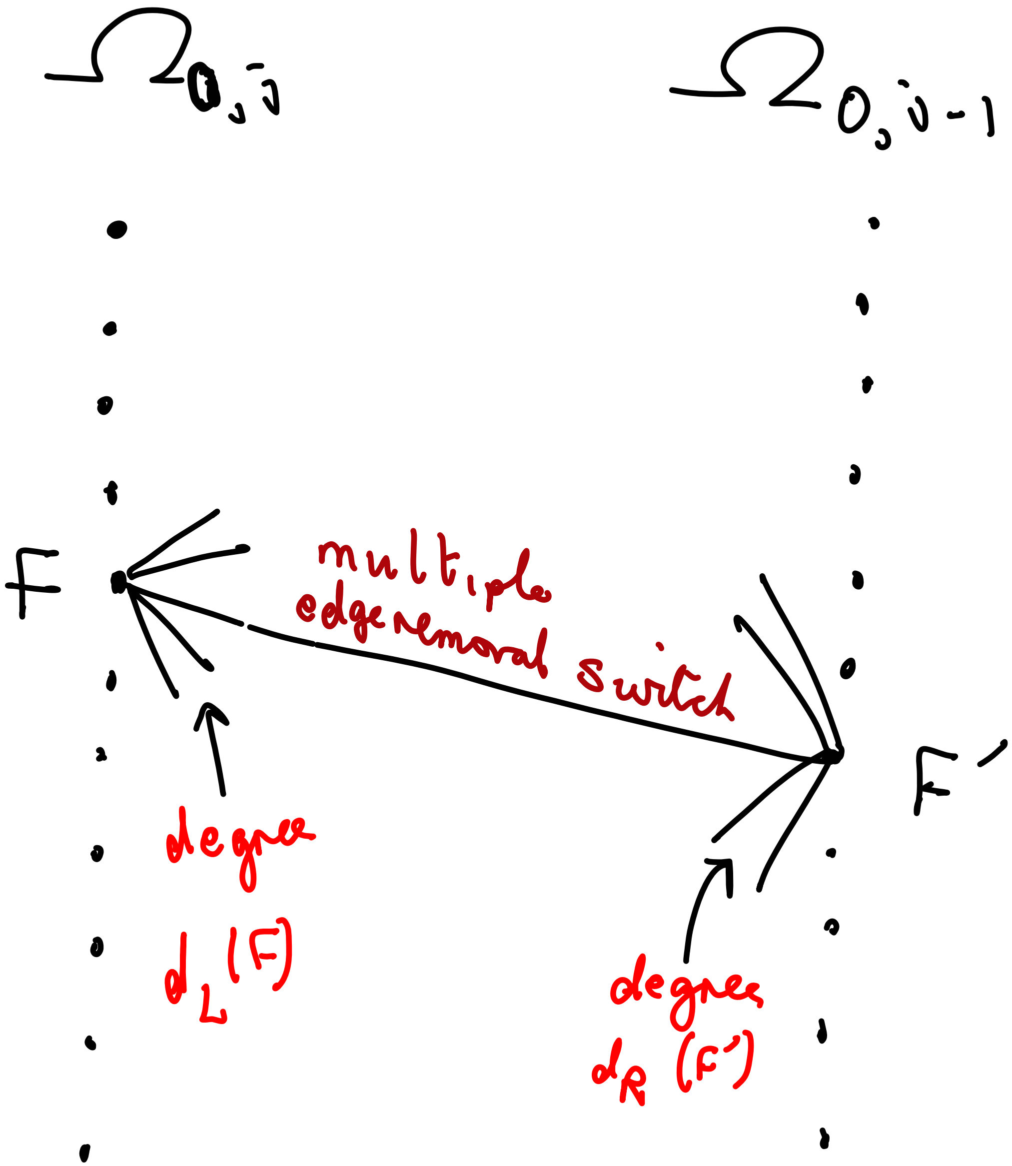
$$\ge i M_1^2 |\Omega_{i,j}| \times \left(1 - \tilde{O}(i\Delta^2/M_1)\right) \qquad\qquad \ge \tfrac{1}{2} M_1 M_2 |\Omega_{i-1,j}| \times \left(1 - \tilde{O}\left(\frac{\Delta^3}{M_1} + \frac{\Delta^3}{M_2}\right)\right)$$

So

$$\frac{|\Omega_{i-1,j}|}{|\Omega_{i,j}|} = \frac{2iM_1}{M_2}\left(1 + \tilde{O}\left(\frac{\Delta^3}{n}\right)\right)$$



In general this operation takes a member F of $\Omega_{i,j}$ to a member F' of $\Omega_{i,j-1}$ unless it creates new loops or multiple edges.



$\Omega_{0,j}$
$\Omega_{0,j-1}$
F
F'
multiple edge removal switch
degree $d_L(F)$
degree $d_R(F')$

choices : $\le j \times M_1^2$

$\ge j \times M_1^2 - \tilde{O}(j M_1 \Delta^2)$

choices $\le M_2^2/4$

$\ge M_2^2/4 - \tilde{O}(M_2 \Delta^3)$

Now

$$\sum_{F \in \Omega_{0,j}} d_L(F) = \sum_{F' \in \Omega_{0,j-1}} d_R(F')$$

$$\le jM_1^2 |\Omega_{0,j}| \qquad\qquad \le \tfrac{1}{4} M_2^2 |\Omega_{0,j-1}|$$

$$\ge jM_1^2 |\Omega_{0,j}| \times \left(1 - \tilde{O}(j\Delta^2/M_1)\right) \qquad\qquad \le \tfrac{1}{4} M_2^2 |\Omega_{0,j-1}| \times \left(1 - \tilde{O}\left(\frac{\Delta^3}{M_2}\right)\right)$$

So

$$\frac{|\Omega_{0,j-1}|}{|\Omega_{0,j}|} = \frac{4jM_1^2}{M_2^2}\left(1 + \tilde{O}\left(\frac{\Delta^3}{n}\right)\right)$$

$\square$

So if $F \overset{\text{ran}}{\in} \Omega$,

$$\Pr(\gamma(F) \text{ is simple}) \approx e^{-\lambda(\lambda+1)}$$

where $\lambda = \dfrac{\sum d_i(d_i-1)}{2\sum d_i}$,

So for any (multi) graph property $\mathcal{P}$

$$\Pr(G_{n,\underline{d}} \in \mathcal{P}) \leq (1+o(1))\, e^{\lambda(\lambda+1)} \Pr(\gamma(F) \in \mathcal{P})$$

assuming $\Delta \leq n^{1/6}$ [Not best known.]

$$\Pr(G_{n,d} \in \mathcal{P}) \leq (1+o(1))\, e^{\lambda(\lambda+1)} \Pr(\gamma(F) \in \mathcal{P})$$

This is particularly useful if $\lambda = O(1)$ e.g. random r-regular graphs where r is a constant. Here $\lambda = \frac{r-1}{2}$.

Theorem

Let $G_{n,r}$ denote a random r-regular graph, $r \geq 3$ constant, vertex set $[n]$. Then whp $G_{n,r}$ is r-connected.

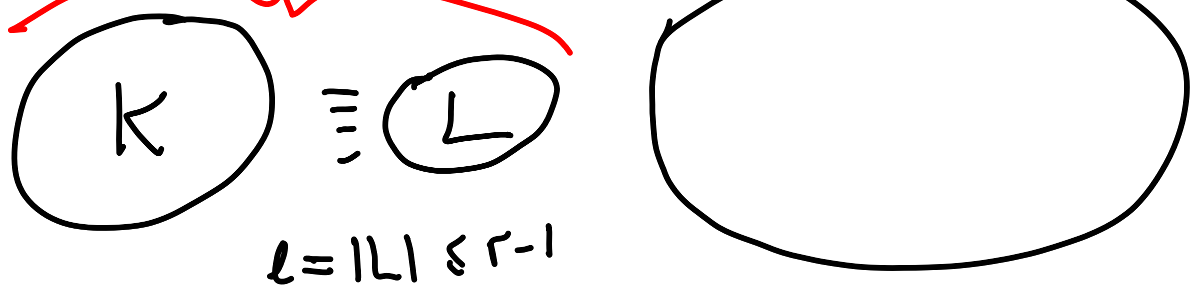
Corollary

If n is even then whp $G_{n,r}$ has a perfect matching.

[An r-edge connected, r-regular graph, with n even, has a perfect matching.]

Proof

$$\#\left\{\text{edges} \subseteq K\cup L \text{ incident with } K\right\} \geq \frac{rk+l}{2}$$



$l = |L| \le r-1$

$L = N(K)$

(i) $\overbrace{\frac{r}{r-2}}^{r_0} \le k = |K| \le n e^{-10}$

$$P(\exists K, L) \le \sum_{k,l} \binom{n}{k}\binom{n}{l}\binom{rk}{\frac{rk+l}{2}}\left(\frac{r(k+l)}{rn}\right)^{\frac{rk+l}{2}}$$

$$\le \sum_{k,l} n^{-(\frac{r}{2}-1)k + l/2} \frac{e^{k+l}}{k^k l^l}\, 2^{rk} (k+l)^{\frac{rk+l}{2}}$$

$$\le \sum_{k,l} n^{-(\frac{r}{2}-1)k + l/2} \frac{e^{k+l}}{k^k l^l} 2^{rk} (k+l)^{\frac{rk+l}{2}}$$

$$\left(\frac{k+l}{l}\right)^{l/2} \le e^{k/2} \qquad \left(\frac{k+l}{k}\right)^{k/2} \le e^{l/2}$$

$$(k+l)^{rk/2} \le k^{rk/2} e^{lr/2}$$

$$\frac{r-1}{2k} < \frac{r}{2} - 1$$

$$k > \frac{r-1}{r-2}$$

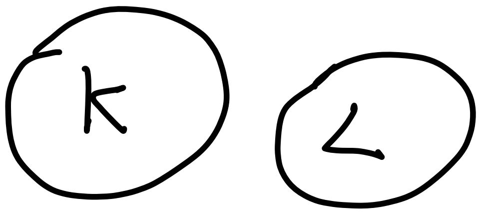
$$\le C_r \sum_{k=r}^{n/10} \sum_{l=0}^{r-1} n^{-(\frac{r}{2}-1)k + l/2} e^{3k/2} k^{(r-2)k/2}$$

constant

$$= C_r \sum_{k=r_0}^{n/10} \sum_{l=0}^{r-1} \left(n^{-(\frac{r}{2}-1) + \frac{l}{2k}} e^{3/2} k^{\frac{r}{2}-1} 2^r \right)^k$$

$$= o(1).$$

(ii) $2 \le k \le \frac{r}{r-2} \le 3$



$K \cup L$ contains

$\frac{rk}{2} + l \ge k + l + 1$ edges

$$\Pr(\exists S : s = |S| \le r - 1 + 3, \text{ contains } s+1 \text{ edges})$$

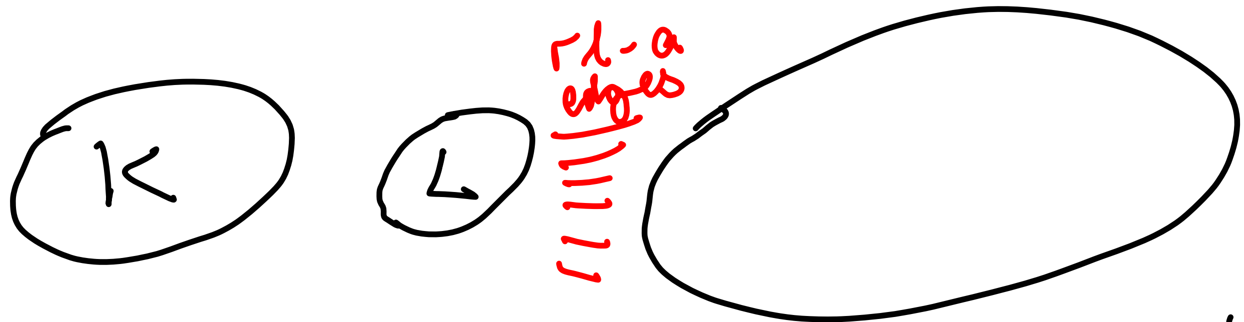
$$\le \sum_{s=4}^{r+2} \binom{n}{s} \binom{rs/2}{s+1} \left(\frac{rs}{rn}\right)^{s+1}$$

$$\le \sum_{s=4}^{r+2} n^{s} \cdot 2^{rs/2} \cdot s^{s+1} \cdot n^{-s-1}$$

$$= o(1).$$

(iii) $ne^{-10} < k \le \frac{n}{2}$

$$\phi(2m) = \frac{(2m)!}{m!\,2^m} \approx 2^{\frac{1}{2}} \left(\frac{2m}{e}\right)^m$$



$$\Pr(\exists K, L) \le \sum_{k,l,a} \binom{n}{k}\binom{n}{l}\binom{rl}{a} \frac{\phi(rk+rl-a)\,\phi(r(n-k-l)+a)}{\phi(rn)}$$

$$\le C_r \sum_{k,l,a} \left(\frac{ne}{k}\right)^k \left(\frac{ne}{l}\right)^l \frac{(rk+rl-a)^{rk+rl-a}\,(r(n-k-l)+a)^{r(n-k-l)+a}}{(rn)^{rn}}$$

$$\le C_r \sum_{k,l,a} \left(\frac{ne}{k}\right)^k \left(\frac{ne}{l}\right)^l e^{O(1)} \left(\frac{k}{n}\right)^{rk} \left(1-\frac{k}{n}\right)^{r(n-k)}$$

$$= C_r \sum_{k,l,a} \left(\frac{ne}{k}\right)^k \left(\frac{ne}{l}\right)^l e^{O(1)} \left(\frac{k}{n}\right)^{rk} \left(1 - \frac{k}{n}\right)^{r(n-k)}$$

$$\leq C_r \sum_{k,l,a} \left(\left(\frac{k}{n}\right)^{r-1} \cdot e^{1-r/2} \cdot n^{r/k} \right)^k$$

$$= O(1).$$

$$\square$$