Expected Length of Minimum Spanning Tree

Let X_e, $e \in E(K_n)$ be a collection of independent uniform $[0,1]$ random variables.

Consider X_e to be the length of edge e.

Let L_n be the length of the minimum spanning tree of K_n.

Theorem

$$\lim_{n\to\infty} E(L_n) = \zeta(3) = \sum_{k=1}^{\infty} \frac{1}{k^3} = 1.202\cdots$$

Proof

Suppose that $T = T(\{X_e\})$ is the minimum spanning tree, unique with probability one.

$$L_n = \sum_{e \in T} X_e$$

$$= \sum_{e \in T} \int_{p=0}^{1} \mathbb{1}_{p \le X_e} \, dp$$

$$a = \int_0^1 \mathbb{1}_{x \le a} \, dx$$

$$= \int_{p=0}^{1} \sum_{e \in T} \mathbb{1}_{p \le X_e} \, dp$$

$$= \int_{p=0}^{1} \sum_{e \in T} \mathbb{1}_{p \leq X_e} \, dp$$

$$= \int_{p=0}^{1} \left| \{ e \in T : X_e \geq p \} \right| dp$$

$$= \int_{p=0}^{1} \left(\kappa(G_p) - 1 \right) dp .$$

$\kappa(G) = \#$ components of G.

$G_p =$ graph induced by edge e with $X_e \leq p$

$\equiv G_{n,p}$

$$E(L_n) = \int_{p=0}^{1} \left(E(\kappa(G_p) - 1 \right) dp .$$

So we estimate $E(k(G_p))$.

(i) $p \geq \frac{6\log n}{n} \implies E(k(G_p)) = 1 + o(1)$.

$$E(k(G_p)) \leq 1 + n\Pr(G_p \text{ is not connected})$$

$$\leq 1 + n\sum_{k=1}^{\frac{1}{2}n} \binom{n}{k} k^{k-2} p^{k-1} (1-p)^{k(n-k)}$$

$$\leq 1 + n^2 \sum_{k=1}^{\frac{1}{2}n} \left(ne \cdot \frac{6\log n}{n} \cdot \frac{1}{n^3} \right)^k$$

$$= 1 + o(1).$$

$$E(L_n) = \int_{p=0}^{\frac{6\log n}{n}} E(\kappa(G_p) - 1)\, dp + O(1)$$

$$= \int_{p=0}^{\frac{6\log n}{n}} E(\kappa(G_p))\, dp + O(1).$$

Write

$$\kappa(G_p) = \sum_{k=1}^{(\log n)^2} A_k + \sum_{k=1}^{(\log n)^2} B_k + C$$

components of size $\geq (\log n)^2$.

of k-components that are trees

of k-components that are not trees

$$E(A_k) = \binom{n}{k} k^{k-2} p^{k-1} (1-p)^{k(n-k) + \binom{k}{2} - k + 1}$$

$$= (1 + o(1)) \; n^k \cdot \frac{k^{k-2}}{k!} \, p^{k-1} (1-p)^{kn}$$

uniform in p

$$E(B_k) \leq \binom{n}{k} k^{k-2} \binom{k}{2} p^k (1-p)^{k(n-k)}$$

$$\leq (1 + o(1)) \left(np \, e^{1-np} \right)^k$$

$$\leq (1 + o(1))$$

$$C \leq \frac{n}{(\log n)^2} \cdot$$

$$\int_{p=0}^{\frac{6\log n}{n}} \sum_{k=1}^{(\log n)^2} E(B_k)\, dp \leq \frac{6\log n}{n} \cdot (\log n)^2 \cdot (1 + o(1))$$

$$= o(1).$$

$$\int_{p=0}^{\frac{6\log n}{n}} C\, dp \leq \frac{6\log n}{n} \cdot \frac{n}{(\log n)^2} = o(1)$$

$$E(L_n) =$$

$$o(1) + (1 + o(1)) \sum_{k=1}^{(\log n)^2} n^k \cdot \frac{k^{k-2}}{k!} \int_{p=0}^{\frac{6\log n}{n}} p^{k-1} (1-p)^{kn}\, dp$$

$$\sum_{k=1}^{(\log n)^2} n^k \cdot \frac{k^{k-2}}{k!} \int_{p=\frac{6\log n}{n}}^{1} p^{k-1} (1-p)^{kn} \, dp$$

$$\leqslant \sum_{k=1}^{(\log n)^2} n^k e^k \int_{p=\frac{6\log n}{n}}^{1} n^{-6k} \, dp \quad = o(1).$$

$$E(L_n) =$$

$$o(1) + (1+o(1)) \sum_{k=1}^{(\log n)^2} n^k \cdot \frac{k^{k-2}}{k!} \int_{p=0}^{1} p^{k-1} (1-p)^{kn} \, dp$$

$$= o(1) + (1 + o(1)) \sum_{k=1}^{(\log n)^2} n^k \cdot \frac{k^{k-2}}{k!} \frac{(k-1)!\,(k(n-k))!}{(k(n-k+1))!}$$

$$= o(1) + (1 + o(1)) \sum_{k=1}^{(\log n)^2} n^k k^{k-3} \prod_{i=1}^{k} \frac{1}{k(n-k)+i}$$

$$= o(1) + (1 + o(1)) \sum_{k=1}^{(\log n)^2} \frac{1}{k^3}$$

$$= o(1) + (1 + o(1)) \sum_{k=1}^{\infty} \frac{1}{k^3}.$$

$\square$