Every monotone property has a threshold

Let $\mathcal{G}$ be a monotone increasing property of graph. Assume $\overline{K_n} \notin \mathcal{G}$ and $K_n \in \mathcal{G}$.

Given $0 < \epsilon < 1$ we define $p(\epsilon)$ by

$$\Pr\left(G_{n, p(\epsilon)} \in \mathcal{G} \right) = \epsilon .$$

$p(\epsilon)$ exists because $\Pr(G_{n,p} \in \mathcal{G})$ is a polynomial in p that increases from 0 $(p=0)$ to 1 $(p=1)$.

Theorem

$p^* = p(\tfrac{1}{2})$ is a threshold for $\mathcal{G}$.

Proof

Suppose $G_1, G_2, \ldots, G_k$ are independent copies of $G_{n,p}$. Then

"same distribution as"

(i) $G_1 \cup G_2 \cup \cdots \cup G_k \cong G_{n,\underbrace{1-(1-p)^k}_{\leq kp}} \leq G_{n,kp}$.

coupling

(ii) With this coupling

$$G_{n,kp} \notin \mathcal{G} \Rightarrow G_1, G_2, \ldots, G_k \notin \mathcal{G}.$$

So

$$\Pr(G_{n,kp} \notin \mathcal{G}) \leq \Pr(G_{n,p} \notin \mathcal{G})^k .$$

(i) Suppose now $p = p^*$ and $k = \omega \to \infty$

$$\Pr(G_{n,\omega p^*} \notin \mathcal{G}) \leq 2^{-\omega} = o(1).$$

(ii) Now suppose $p = p^*/\omega$.

$$\tfrac{1}{2} = \Pr(G_{n,p^*} \notin \mathcal{G}) \leq \Pr(G_{n,p^*/\omega} \notin \mathcal{G})^{\omega}$$

So $$\Pr(G_{n,p^*/\omega} \notin \mathcal{G}) \geq 2^{-1/\omega} = 1 - o(1).$$

$\square$