Small Subgraphs

Let H be a fixed graph.

We use the notation n_H, e_H for the number of vertices and edges of G.

Also let

$$\rho_H = \frac{e_H}{v_H}.$$

Lemma

Let X_H denote the number of copies of H in $G_{n,p}$.

$$E(X_H) = \binom{n}{v_H} \frac{v_H!}{aut(H)} p^{e_H}$$

where

$aut(H)$ is the number of automorphisms of H.

Proof

K_n contains $\binom{n}{v_H} a_H$ distinct copies of H, where a_H is the number of copies of H in K_{v_H}. Thus

$$E(X_H) = \binom{n}{v_H} a_H p^{e_H}$$

and all we need to show is that

$$a_H \times \operatorname{aut}(H) = v_H!$$

Each permutation σ of $[v_H]$ defines a unique copy of H as follows:

A copy of H corresponds to a set of e_H edges of K_{v_H}. The copy H_σ corresponding to σ has edges $\{(x_{\sigma(i)}, y_{\sigma(i)}) : 1 \leq i \leq e_H\}$ where $\{(x_i, y_i) : 1 \leq i \leq e_H\}$ is some fixed copy of H in K_{v_H}.

But $H_\sigma = H_{\tau\sigma}$ iff $\forall i\ \exists j$ such that $(x_{\tau\sigma(i)}, y_{\tau\sigma(i)}) = (x_{\sigma(j)}, y_{\sigma(j)})$ i.e. τ is an automorphism of H. $\square$

Theorem

Suppose $p = o(n^{-1/\rho_H})$. Then whp, $G_{n,p}$ contains no copies of H.

Proof

Suppose that $p = \frac{1}{\omega} n^{-1/\rho_H}$ where $\omega(n) \to \infty$. Then

$$E(X_H) \leq n^{v_H} \omega^{-e_H} n^{-e_H/\rho_H}$$

$$= \omega^{-e_H}$$

$$\to 0.$$

$\square$

Now consider the case where $n^{1/\rho_H} p \to \infty$.

Suppose $p = \omega n^{-1/\rho_H}$ where $\omega \to \infty$.

Then for some constant $c_H > 0$

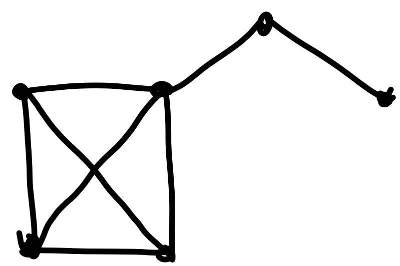
$$\mathbb{E}(X_H) \geq c_H n^{v_H} \omega^{e_H} n^{-e_H/\rho_H}$$

$$= c_H \omega^{e_H}$$

$$\to \infty.$$

This is not enough to show that whp $G_{n,p}$ contains a copy of H.

Suppose

$H =$ 

$n_H = 6$

$e_H = 8$

Let $p = n^{-5/7}$. Then $1/\rho_H > \frac{5}{7}$ and so

$$E(X_H) \approx c_H n^{6 - 8 \times 5/7} \longrightarrow \infty.$$

On the other hand, if $\hat{H} = K_4$ then

$$E(X_{\hat{H}}) \leq n^{4 - 6 \times 5/7} \longrightarrow 0$$

and so whp there are no copies of $\hat{H}$ and hence no copies of H.

Theorem

Let $\rho^*_H = \max_{\substack{H' \subseteq H \\ v_{H'} > 0}} \rho_{H'}$.

(a) If $n^{-1/\rho^*_H} p \longrightarrow 0$

then whp $X_H = 0$.

(b) If $n^{-1/\rho^*_H} p \longrightarrow \infty$

then whp $X_H > 0$.

Proof

(a) follows from p5 because in this case there is whp, an $H' \subseteq H$ such that $X_{H'} = 0$.

(b) We use the second moment method:

$$Pr(X_H > 0) \geq \frac{E(X_H)^2}{E(X_H^2)}$$

$$E(X_H^2) = \sum_{i,j=1}^{N_H} P_r(H_i \wedge H_j)$$

$H_1, H_2, \ldots\ldots$ are all copies of H in K_n.

$$= E(X_H) \sum_{j=1}^{N_H} P_r(H_j \mid H_1)$$

$$\leq E(X_H)^2 + E(X_H) \sum_{\substack{H' \subseteq H \\ H' \neq H \\ e_{H'} > 0}} n^{v_H - v_{H'}} \, p^{e_H - e_{H'}}$$

H

So

constant

$$\frac{E(X_H^2)}{E(X_H)^2} - 1 \leq c_H \sum_{\substack{H' \subseteq H \\ H' \neq H \\ e_{H'} > 0}} n^{-v_{H'}} \, p^{-e_{H'}}$$

But

$$\sum_{\substack{H' \subseteq H \\ H' \neq H}} n^{-v_{H'}} p^{-e_{H'}} = \sum_{\substack{H' \subseteq H \\ H' \neq H \\ e_{H'} > 0}} n^{e_{H'}\left(\frac{1}{\rho_H^*} - \frac{1}{\rho_{H'}}\right)} \omega^{-e_{H'}}$$

$p = \omega n^{-1/\rho_H^*}$

$$= O(\omega^{-1}).$$

Thus

$$\frac{E(X_H^2)}{E(X_H)^2} = 1 + o(1).$$

$\square$