Small Subgraphs

Let \( H \) be a fixed graph. We use the notation \( \eta_{H,j}, e_{H,j} \) for the number of vertices and edges of \( A \).

Also let

\[
\rho_H = \frac{e_H}{\eta_H}.
\]
Lemma

Let $X_H$ denote the number of copies of $H$ in $G_{n,p}$.

$$E(X_H) = \left( \binom{n}{ \nu_H} \right) \frac{\nu_H!}{\text{aut}(H)} \rho^e_H$$

where

$\text{aut}(H)$ is the number of automorphisms of $H$. 
**Proof**

$K_n$ contains $\binom{n}{v_H} a_H$ distinct copies of $H$, where $a_H$ is the number of copies of $H$ in $K_{v_H}$. Thus

$$E(X_H) = \binom{n}{v_H} a_H^e H$$

and all we need to show is that

$$v_H \times \text{aut}(H) = v_H!$$
Each permutation $\sigma$ of $[\nu_H]$ defines a unique copy of $H$ as follows:

A copy of $H$ corresponds to a set $\Theta$ of edges of $K_{\nu_H}$. The copy $H_\sigma$ corresponding to $\sigma$ has edges $\{(x_{\sigma(i)}, y_{\sigma(i)}): 1 \leq i \leq e_H\}$ where $\Theta(x_i, y_i): 1 \leq i \leq e_H$ is some fixed copy of $H$ in $K_{\nu_H}$.

But $H_\sigma = H_{\tau_\sigma}$ iff $\forall i \exists j$ such that $(x_{\tau_\sigma(i)}, y_{\tau_\sigma(i)}) = (x_{\sigma(i)}, y_{\sigma(i)})$ i.e. $T$ is an automorphism of $H$. \qed
Theorem
Suppose $\rho = o(n^{-1/\rho_H})$. Then $\text{whp}, G_{n,\rho}$ contains no copies of $H$.

Proof
Suppose that $\rho = \frac{1}{w} n^{-1/\rho_H}$ where $w(n) \to \infty$. Then

$$E(X_H) \leq n^{\nu_H} w^{-\epsilon_H} n^{-\epsilon_H/\rho_H}$$

$$= w^{-\epsilon_H}$$

$$\to 0.$$
Now consider the case where $n^{1/\rho_n} p \to \infty$.

Suppose $p = \omega n^{-1/\rho_n}$ where $\omega \to \infty$.

Then for some constant $c_n > 0$

$$E(X_n) \geq c_n n^{1/\rho_n} \omega n^{-e_n/\rho_n}$$

$$= c_n \omega^{e_n/\rho_n}$$

$$\to \infty.$$

This is not enough to show that w.h.p

$C_{n,p}$ contains a copy of $H$.
Suppose

\[ H = \begin{array}{c}
\text{Diagram of graph} \\
\end{array} \]

\[ n_H = 6 \]

\[ e_H = 8 \]

Let \( p = n^{-5/7} \). Then \( 1/p_H > \frac{5}{7} \) and so

\[ E(X_H) = c_H n \]

\[ 6 - 8 \times 5/7 \]

\[ \rightarrow \infty. \]

On the other hand, \( \sqrt{H} = K_4 \), then

\[ E(X_{\hat{H}}) \leq n^{4 - 6 \times 5/7} \]

\[ \rightarrow 0 \]

and so why there are no copies of \( \hat{H} \) and hence no copies of \( H \).
Theorem

Let \( \rho_H^* = \max_{H', \leq H} \rho_{H'} \).
\( \nu_{H'} > 0 \)

(a) If \( n^{1/\rho_H^*} \rho \to 0 \)
then w.p. \( X_H = 0 \).

(b) If \( n^{1/\rho_H^*} \rho \to \infty \)
then w.p. \( X_H > 0 \).
Proof

(a) follows from ps because in this case there is why, an $H' \leq H$ such that $X_{H'} = 0$.

(b) We use the second moment method:

$$
Pr(X_H > 0) \geq \frac{E(X_H^2)}{E(X_H^2)}
$$
\[ E(X_H^2) = \sum_{i,j=1}^{N_H} \rho_i(H_i \land H_j) \]
\[ = E(X_H) \sum_{j=1}^{N_H} \rho_j(H_j \mid H_1) \]
\[ \leq E(X_H)^2 + E(X_H) \sum \sum_{H' \leq H} \sum_{H' \neq H} n_{H-H'} \rho_{e_{H-H'}} \]

So,
\[ \frac{E(X_H^2)}{E(X_H)^2} - 1 \leq c_H \sum \sum_{H' \leq H} \sum_{H' \neq H} n_{H-H'} \rho_{e_{H-H'}} \]

H,, H,, ... are all copies of H in \( \mathcal{K}_n \).
But

\[ \sum_{H' \in H} n^{-e_{H'}} \rho - e_{H'} = \sum_{H' \leq H, \ H' \notin H} e_{H'} \left( \frac{1}{\rho_H^*} - \frac{1}{\rho_{H'}} \right) \omega^{-e_{H'}} \]

\[ \rho = \omega n^{-1/\rho_H^*} \]

= \( O(\omega^{-1}) \).

Thus

\[ \frac{E(X_H^2)}{E(X_H)^2} = 1 + O(1). \]