Independence and Chromatic Number

Theorem

Suppose $0 < p < 1$ is constant and $b = \frac{1}{1-p}$.

Then whp

$$\alpha(G_{n,p}) \approx 2\log_b n$$

$\alpha(G)$ = size of largest independent set in G.

Proof

Let X_k = # of independent sets of size k.

(1) Let $k = \lceil 2\log_b n \rceil$

$$E(X_k) = \binom{n}{k} (1-p)^{\binom{k}{2}}$$

$$\leq \left(\frac{ne}{k(1-p)^{1/2}} \cdot (1-p)^{k/2} \right)^k$$

$$\leq \left(\frac{e}{k(1-p)^{1/2}} \right)^k$$

$$= O(1).$$

(ii) Let now

$$k = \lfloor 2\log_b n - 3\log_b \log_b n \rfloor$$

Let $\Delta^* = \sum_{\substack{i,j \\ S_i \sim S_j}} \Pr(S_i \wedge S_j \text{ are independent in } G_{n,p})$

where $S_1, S_2, \ldots S_{\binom{n}{k}}$ are all k-subsets of $[n]$

and $S_i \sim S_j$ iff $|S_i \cap S_j| \geq 2$.

$$\Pr(X_k = 0) \leq \exp\left\{ - \frac{E(X_k)^2}{\Delta^*} \right\}$$

Janson's Inequality

$$\frac{\Delta^*}{E(X_k)^2} = \frac{\binom{n}{k}(1-p)^{\binom{k}{2}} \sum_{j=2}^{k} \binom{n-k}{k-j}\binom{k}{j}(1-p)^{\binom{k}{2}-\binom{j}{2}}}{\left(\binom{n}{k}(1-p)^{\binom{k}{2}}\right)^2}$$

$$= \sum_{j=2}^{k} \underbrace{\frac{\binom{n-k}{k-j}\binom{k}{j}}{\binom{n}{k}}(1-p)^{-\binom{j}{2}}}_{u_j}$$

$$\frac{u_j}{u_2} \le \left[\frac{k}{n-2k}\cdot\frac{ke}{j-2}\cdot(1-p)^{-\frac{j+1}{2}}\right]^{j-2} \qquad j>2$$

$$< 1.$$

So

$$\frac{E(X_k)^2}{\Delta^*} \geq \frac{1}{k u_2} \geq \frac{n^2(1-p)}{k^5}$$

So

$$\Pr(X_k = 0) \leq e^{-\Omega(n^2/(\log n)^5)}, \quad (*)$$

$\square$

Theorem

$$\chi(G_{n,p}) \approx \frac{n}{2\log_b n}$$

Proof

(1)

$$\chi(G_{n,p}) \geq \frac{n}{\alpha(G_{n,p})}$$

$$\approx \frac{n}{2\log_b n}.$$

(ii)

Let $\nu = \frac{n}{(\log n)^2}$. It follows from (*)

on p5 that

$$\Pr(\exists S : |S| \geq \nu \text{ and } S \not\supseteq \text{independent set of size } \geq k_0 = 2\log_b n - 3\log_b \log_b n)$$

$$\leq \binom{n}{\nu} \exp\left\{-\Omega\left(\frac{\nu^2}{(\log n)^5}\right)\right\}$$

$$= o(1).$$

So assume that every set of size $\geq \nu$ contains an independent set of size $\geq k_0$.

So we repeatedly

Choose an independent set of size k_0.

Give it a new colour.

Repeat until number of uncoloured vertices is $\leq \nu$.

Give each remaining vertex its own colour.

Number of colours used

$$\leq \frac{n}{k_0} + \nu \approx \frac{n}{2\log_b n}.$$

$\square$

Performance of Greedy Algorithm

Algorithm (GREEDY)

k is current colour

A is current set of vertices that might get color k in current round.

U is current set of uncoloured vertices.

begin

$k \leftarrow 0$; $A \leftarrow [n]$; $U \leftarrow [n]$;

while $U \neq \emptyset$;

begin

START ITERATION

$k \leftarrow k+1$; $A \leftarrow U$;

Chose $v \in A$ and give it colour k;

$U \leftarrow U \setminus \{v\}$

$A \leftarrow A \setminus (\{v\} \cup N(v))$

If $A \neq \emptyset$

otherwise

end

Theorem

Whp GREEDY uses $\approx \dfrac{n}{\log_b n}$ colours

(about twice as many as it "should").

Proof

At the start of an iteration the edges inside U are unexamined. Suppose that $|U| \geq \nu = \dfrac{n}{(\log n)^2}$. We show that $\approx \log_b n$ vertices get colour k.

Each iteration choose a maximal independent set from the remaining uncolored vertices.

$$\Pr\Big(\exists S : |S| \le \underbrace{\log_b n - 3\log_b \log_b n}_{k_0} \text{ and } S \text{ is maximal independent}\Big)$$

$$\le \sum_{s=1}^{k_0} \binom{n}{s} (1-p)^{\binom{s}{2}} \left(1-(1-p)^s\right)^{n-s}$$

$$\le \sum_{s=1}^{k_0} \left[\frac{ne}{s} (1-p)^{\frac{s-1}{2}} \right]^s e^{-(n-s)(1-p)^s}$$

$$\le \sum_{s=1}^{k_0} \left(n e^{1+(1-p)^s} (1-p)^{\frac{s-1}{2}} \right)^s e^{-n(1-p)^s}$$

$$\leq \sum_{s=1}^{k_0} \left(n e^{1+(1-p)^s} (1-p)^{\frac{s-1}{2}} \right)^s e^{-n(1-p)^s}$$

$$\leq \sum_{s=1}^{k_0} \left(n e^2 \right)^s e^{-n(1-p)^s}$$

$$\leq k_0 (ne^2)^{k_0} e^{-(\log_b n)^3}$$

$$\leq e^{-(\log_b n)^3/2}.$$

So the probability that we fail to use $\geq k_0$ colours while $|U| \geq \nu$ is at most $n e^{-(\log_b \nu)^3/2}$

$= o(1).$

On the other hand let

$$k_1 = \log_b n + 2\log_b \log_b n$$

Consider one round. Let $U_0 = V$ and suppose $u_1, u_2, \ldots$ get colour k and $U_{i+1} = U_i \setminus (\{u_i\} \cup N(u_i))$.

Then $$E(|U_{i+1}| \mid U_i) \leqslant |U_i|(1-p)$$

and so $$E(|U_k|) \leqslant n(1-p)^k.$$

So $$\Pr(k_1 \text{ vertices coloured in a round}) \leqslant \frac{1}{(\log_b n)^2}$$

$$\Pr(2k_1 \text{ vertices coloured in a round}) \leqslant \frac{1}{n^3}$$

So let

$$\delta_i = \begin{cases} 1 & \leq k_1 \text{ colours used in Round } i \\ 0 & \text{otherwise} \end{cases}$$

We see that

$$\Pr(\delta_i = 1 \mid \delta_1, \delta_2, \dots \delta_{i-1}) = 1 - O(1/(\log n)^2)$$

and deduce that whp $O(n/(\log_b n)^2)$ rounds colour more than k_1 vertices and no round colours more than $2k_1$ vertices.

$\square$

Concentration

Theorem

$$\Pr\left(\left|X(G_{n,p}) - E(X(G_{n,p}))\right| \geq t\right) \leq 2e^{-\frac{t^2}{2n}}$$

Proof

Write $X = Z(Y_1, Y_2, \ldots, Y_n)$ where

$$Y_j = \{ (i, j) \in E(G_{n,p}) : i < j \}.$$

Then
$$|Z(Y_1, \ldots, Y_i, \ldots, Y_n) - Z(Y_1, \ldots, \hat{Y}_i, \ldots, Y_n)| \leq 1$$

and the theorem follows from a martingale inequality.

$\square$