The diameter of Random Graphs

Theorem

Let $d \geq 2$ be a fixed positive integer. Suppose that $c > 0$ and

$$p^d n^{d-1} = \ln(n^2/c).$$

Then

$$\lim_{n \to \infty} \Pr(\text{diameter } G_{n,p} = k) = \begin{cases} e^{-c/2} & : k = d \\ 1 - e^{-c/2} & : k = d+1. \end{cases}$$

(a) Whp $\mathrm{diam}(G) \geq d$.

Fix $v \in V$ and let

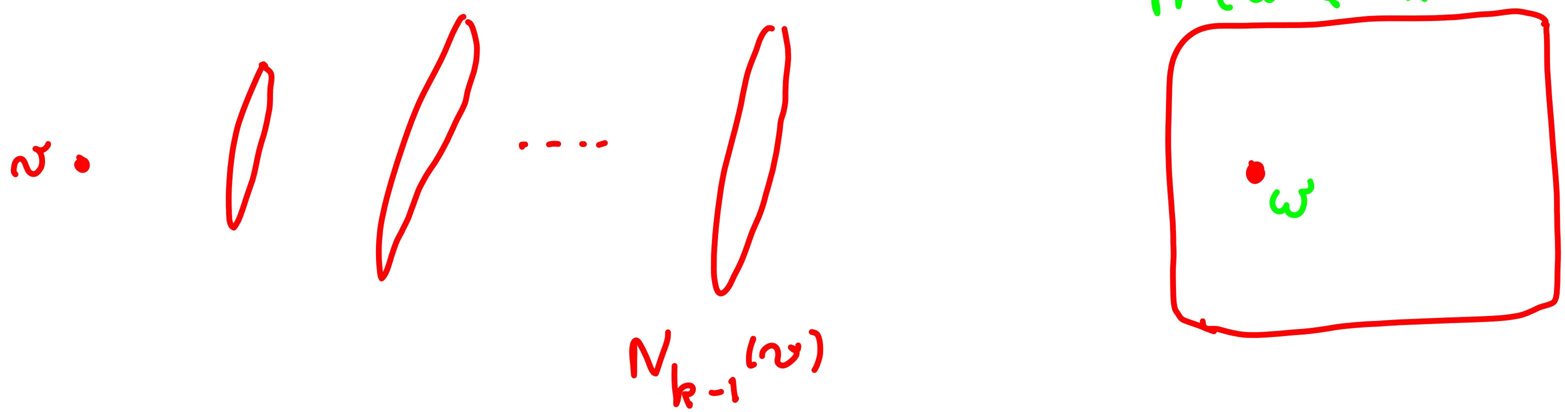
$$N_k(v) = \{ w : \mathrm{dist}(v, w) = k \}.$$

We show that whp, for $0 \leq k < d$,

$$\begin{aligned} |N_k(v)| &\leq (2np)^k \\ &\approx (2n \ln n)^{k/d} \\ &= o(n). \end{aligned}$$

We observe that given $N_i(v)$, $i = 0, 1, \ldots, k-1$, that $|N_k(v)|$ is distributed as

$$\mathrm{Bin}\left(n - \sum_{i=0}^{k-1} |N_i(v)| \, , \quad 1 - \underbrace{(1-p)^{|N_{k-1}(v)|}}_{\Pr(w \notin N_k(v))} \right)$$



Let $\mathcal{E}_i = \{ |N_i(v)| \leq (2np)^i \}$.

Condition on $\mathcal{E}_0, \mathcal{E}_1, \ldots \mathcal{E}_{k-1}$.

Does not condition on edges from $N_{k-1}(v)$ to $V \setminus \bigcup_{i=0}^{k-1} N_i(v)$.

$|N_k(v)|$ is distributed as $Bin(\nu, q)$

where $\nu < n$ and

$$q = 1 - (1-p)^{|N_{k-1}(v)|}$$

$$\leq |N_{k-1}(v)| p.$$

$$\leq (2np)^{k-1} p$$

$$< 1.$$

Thus

$$E\left(|N_k(v)| \;\middle|\; \mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_{k-1} \right)$$

$$= vq$$

$$\le np\,|N_{k-1}(v)|.$$

Chernoff bound gives

$$\Pr\left(|N_k(v)| \ge (2np)^k \;\middle|\; \mathcal{E}_0, \mathcal{E}_1, \dots, \mathcal{E}_{k-1} \right)$$

$$\le \Pr\left(\mathrm{Bin}(n, |N_{k-1}(v)|p) \ge (2np)^k \;\middle|\; \mathcal{E}_{k-1} \right)$$

$$\le \Pr\left(\mathrm{Bin}(n, (2np)^{k-1}p) \ge (2np)^k \right)$$

$$\leq \Pr\left(\mathrm{Bin}\left(n, (2np)^{k-1} p\right) \geq (2np)^{k}\right)$$

$$\leq e^{-(2np)^{k-1} np/3}$$

$$\ll n^{-2}.$$

So

$$\Pr\left(\bigcup_{i=0}^{d-1} N_i(v) = [n]\right) \leq$$

$$\sum_{k=1}^{d-1} \Pr\left(\overline{\mathcal{E}_k} \mid \mathcal{E}_1, \dots \mathcal{E}_{k-1}\right) =$$

$$O(n^{-2}).$$

(b) Whp $\operatorname{diam}(G) \leq d+1$.

Proof

Fix $v, w \in [n]$. Then for $1 \leq k < d$,

Let $\mathcal{F}_k = \left\{ |N_k(v)| \geq \left(\frac{np}{2}\right)^k \right\}$.

$$\Pr\left(\overline{\mathcal{F}_k} \mid \mathcal{E}_1, \mathcal{F}_1, \ldots, \mathcal{E}_{k-1}, \mathcal{F}_{k-1}\right) =$$

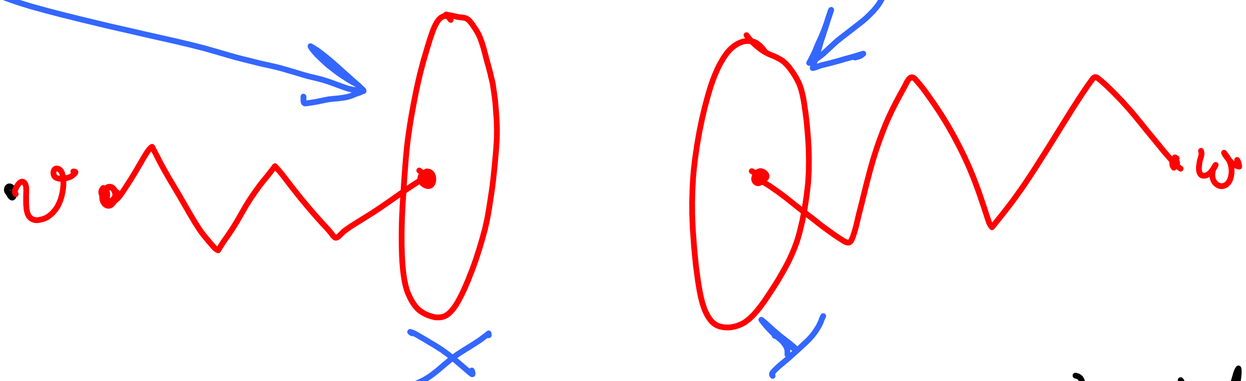
$$\Pr\left(\operatorname{Bin}\Big(\underbrace{n - \sum_{i=0}^{k-1} |N_i(v)|}_{n - o(n)},\ \underbrace{1 - (1-p)^{|N_{k-1}(v)|}}_{\geq \frac{3}{4} \cdot \left(\frac{np}{2}\right)^{k-1} p}\Big) \leq \left(\frac{np}{2}\right)^k\right)$$

$$\leq e^{-\Omega\left(\left(\frac{np}{2}\right)^k\right)} = O(n^{-3}).$$

So with probability $1 - O(n^{-3})$,

$$|N_{\lfloor d/2\rfloor}(v)| \ge \left(\frac{np}{2}\right)^{\lfloor d/2\rfloor} \quad \text{and}$$

$$|N_{\lceil d/2\rceil}(w)| \ge \left(\frac{np}{2}\right)^{\lceil d/2\rceil}.$$



Either $X \cap Y \neq \emptyset$ and $\mathrm{dist}(v, w) \le \lfloor d/2 \rfloor + \lceil d/2 \rceil = d$

or

$$\Pr(\nexists\, X{:}Y \text{ edge}) \le (1-p)^{\left(\frac{np}{2}\right)^d}$$

$$\Pr(\nexists X : Y \text{ edge}) \leq (1-p)^{\left(\frac{np}{2}\right)^d}$$

$$\leq \exp\left\{-\left(\frac{np}{2}\right)^d p\right\}$$

$$\leq \exp\left\{-(2-o(1))\, np \ln n\right\}$$

$$= o(n^{-3}).$$

So

$$\Pr(\exists v, w : \text{dist}(v, w) > d+1) = o(n^{-1}).$$

$$p^d n^{d-1} = \ln(n^2/c).$$

Now consider d or $d+1$ as diameter.

We use Janson's inequality.

For $v, w \in [n]$ let

$$\mathcal{A}_{v,w} = \{ v, w \text{ are not joined by a path of length } d \}$$

For $x_1, x_2, \ldots, x_{d-1}$ let

$$\mathcal{B}_{\underbrace{x_1, x_2, \ldots, x_{d-1}}_{x}} = \{ (v, x_1, x_2, \ldots, x_{d-1}, w) \text{ is a path in } G_{n,p} \}.$$

Let $$Z = \sum_{x = x_1, \dots, x_{d-1}} Z_x \leftarrow \begin{cases} 1 : B_x \text{ occurs} \\ 0 : \neg B_x \text{ occurs} \end{cases}$$

$$\mu = E(Z) = (n-2)(n-3)\dots(n-d)\,p^d$$
$$\approx \ln(n^2/c).$$

Let
$$\Delta = \sum_{\substack{x = x_1, x_2, \dots x_{d-1} \\ y = y_1, y_2, \dots y_{d-1}}} Pr(B_x \cap B_y)$$

v, x, w and v, y, w share an edge

$x \neq y$

$$\leq \sum_{c=1}^{d-1} n^{2(d-1)-c} \, p^{2d-c}$$

edges in common between x and y

$$= O\left(\sum_{c=1}^{2d-2} n^{2(d-1)-c-\frac{d-1}{d}(2d-c)} (\log n)^{\frac{2d-c}{d}}\right)$$

$$= \tilde{O}\left(n^{-c/d}\right)$$

$$= o(1).$$

Applying $\Pr(Z=0) \leq e^{-\mu+\Delta}$ we get

$$\Pr(Z=0) \leq (1+o(1)) \frac{c}{n^2}.$$

On the other hand the FKG inequality implies

$$\Pr(Z=0) \geq (1-p^d)^{(n-2)(n-3)\cdots(n-d)}$$

$$= (1-o(1)) \frac{c}{n^2}$$

So

$$\Pr(A_{v,w}) = \Pr(Z=0) = (1+o(1)) \frac{c}{n^2}.$$

So $\quad E(\# v,w : A_{v,w} \text{ occurs}) \approx c/2$

and we should expect that

$$\Pr(\nexists\, v,w : A_{v,w} \text{ occurs}) \approx e^{-c/2}. \quad (1)$$

Indeed, if we choose $v_1, w_1, v_2, w_2, \dots, v_k, w_k$, k constant, we find that

$$\Pr(A_{v_1,w_1} \cap A_{v_2,w_2} \cap \dots \cap A_{v_k,w_k}) \quad (2)$$

$$\approx \left(\frac{c}{n^2}\right)^k$$

and (1) follows as in previous arguments.

For (2) we define

$$Z = Z_1 + Z_2 + \cdots + Z_k$$

where

$$Z_i = \# \text{ paths of length } d \text{ from } v_i \text{ to } w_i.$$

We need to show that the corresponding $\Delta = o(1)$ and then we need to show that $1 \le r < s \le k$

$$\Delta_{r,s} = \sum_{\substack{x = x_1, x_2, \ldots x_{d-1} \\ y = y_1, y_2, \ldots y_{d-1}}} \Pr\left(\mathcal{B}_x^r \cap \mathcal{B}_y^s\right) = o(1)$$

v_r, x, w_r and v_s, y, w_s share an edge

$x \neq y$

But

$$\Delta_{r,s} \leq \sum_{c=1}^{d-1} n^{2(d-1)-c} \, p^{2d-c}$$

$$= o(1)$$

as before.

$\square$