Janson's Inequality

Suppose that $0 \leq p_i \leq 1$ for $i = 1, 2, \ldots, M$.

Let X be a random subset of $[M]$ where

$$\Pr(i \in X) = p_i$$

independently for $i \in [M]$

Let $S_1, S_2, \ldots, S_\ell \subseteq [M]$ and let

$$Z_i = \begin{cases} 1 & S_i \subseteq X \\ 0 & S_i \nsubseteq X \end{cases}$$

Let $Z = Z_1 + Z_2 + \cdots + Z_\ell$

Count the number of S_i that occur.

Let

$$\mu = E[Z] = \sum_{i=1}^{\ell} p_i$$

and

$$\Delta^* = \frac{1}{2} \sum_{S_i \cap S_j \neq \emptyset} E(Z_i Z_j).$$

<u>Theorem</u>

$$\Pr(Z \leq \mu - t) \leq e^{-t^2/2\Delta^*}.$$

Proof

Let $\psi(s) = E(e^{-sZ})$, $s \geq 0$.

$$\Pr(Z \leq \mu - t) = \Pr(e^{s(\mu - t - Z)} \geq 1)$$

$$\leq E(e^{s(\mu - t - Z)})$$

$$= e^{s(\mu - t)} \psi(s).$$

Write

$$\log \Pr(Z \le \mu - t) \le \log \psi(s) + s(\mu - t)$$

TO BE SHOWN $$\le -\frac{\mu^2}{\Delta^*}\left(1 - e^{-s\Delta^*/\mu}\right) + s(\mu - t).$$

If $t = \mu$ we simply let $s \to \infty$. Otherwise

to minimise RHS we take

$$s = -\frac{\mu}{\Delta^*} \log\left(1 - \frac{t}{\mu}\right)$$

and then

$$\log \Pr(Z \le \mu - t) \le -\frac{\mu}{\Delta^*}\left(t + (\mu - t)\log\left(1 - \frac{t}{\mu}\right)\right)$$

$$\log \Pr(Z \leq \mu - t) \leq -\frac{\mu}{\Delta^*}\left(t \cdot (\mu-t)\log\left(1-\frac{t}{\mu}\right)\right)$$

$$\leq -\frac{\mu}{\Delta^*}\left(t - (\mu-t)\frac{t}{\mu}\right)$$

$$= -\frac{t^2}{2\Delta^*},$$

TO BE SHOWN

$$\log \Psi(s) \leq -\frac{\mu^2}{\Delta^*}\left(1-e^{-s\Delta/\mu}\right).$$

Now

$$\log \psi(s) = \int_{u=0}^{s} \frac{\psi'(u)}{\psi(u)}\, du$$

and

$$\psi'(u) = -E\left(Ze^{-uZ}\right)$$

$$= -\sum_{i=1}^{M} E\left(Z_i e^{-uZ}\right).$$

For each $i \in [M]$ we write

$$Z = X_i + Y_i$$

where

$$X_i := \sum_{j : S_j \cap S_i \neq \emptyset} Z_j .$$

Then

$$E(Z_i e^{-uZ}) = p_i E(e^{-uX_i} e^{-uY_i} \mid Z_i = 1)$$

FKG inequality $\geq p_i E(e^{-uX_i} \mid Z_i = 1) E(e^{-uY_i} \mid Z_i = 1)$

$$= p_i E(e^{-uX_i} \mid Z_i = 1) E(e^{-uY_i})$$

$$= p_i\, E(e^{-uX_i} \mid Z_i = 1)\, E(e^{-uY_i})$$

$$\geq p_i\, E(e^{-uX_i} \mid Z_i = 1)\, \psi(u).$$

So

$$\frac{\psi'(s)}{\psi(s)} \leq -\mu \sum_{i=1}^{M} \frac{p_i}{\mu} E(e^{-sX_i} \mid Z_i = 1)$$

$$\leq -\mu \sum_{i=1}^{M} \frac{p_i}{\mu} e^{-E(sX_i \mid Z_i = 1)}$$

Jensen

$$\leq -\mu \exp\left\{ -\sum_{i=1}^{M} E\left(\frac{s p_i}{\mu} X_i \,\middle|\, Z_i = 1 \right) \right\}$$

$$= -\mu \exp\left\{ -\frac{s}{\mu} \sum_{i=1}^{M} E(X_i \mid Z_i = 1) P(Z_i = 1) \right\}$$

$$= -\mu \exp\left\{ -\frac{s}{\mu} \sum_{i=1}^{M} E(X_i \mid Z_i = 1) P(Z_i = 1) \right\}$$

$$= -\mu e^{-s\Delta^*/\mu}$$

$$\sum_{i=1}^{M} E(X_i \mid Z_i = 1) P(Z_i = 1)$$

$$= \sum_{i=1}^{M} \sum_{j : S_j \cap S_i \neq \emptyset} P(Z_j = 1 \mid Z_i = 1) P(Z_i = 1)$$

$$= \Delta^* .$$

So
$$\frac{\psi'(s)}{\psi(s)} \leqslant -\mu e^{-s\Delta^*/\mu}.$$

$$\log \psi(s) = \int_{u=0}^{s} \frac{\psi'(u)}{\psi(u)}\,du$$

$$\leqslant -\int_{u=0}^{s} \mu e^{-u\Delta^*/\mu}\,du$$

$$= -\frac{\mu^2}{\Delta^*}\left(1 - e^{-s\Delta^*/\mu}\right). \qquad \square$$

Now let

$$\Delta = \frac{1}{2} \sum_{\substack{S_i \cap S_j \neq \emptyset \\ S_i \neq S_j}} E(Z_i Z_j).$$

Theorem

(i) $Pr(Z=0) \leq e^{-\mu+\Delta}$

(ii) $Pr(Z=0) \leq e^{-\frac{\mu^2}{\mu+\Delta}}$

((ii) follows directly from first theorem.)

For (1) we have (see p 8)

$$\frac{\psi'(s)}{\psi(s)} \leq -\sum_i p_i E\left(e^{-sX_i} \mid z_i = 1\right).$$

so

$$\log\big(P(Z=0)\big) = \int_0^\infty \big(\log \psi(s)\big)' \, ds$$

$$\leq -\int_0^\infty \sum_i p_i E\left(e^{-sX_i} \mid Z_i = 1\right) ds$$

$$= -\sum_i p_i \int_{s=0}^{\infty} E\left(e^{-sX_i} \mid Z_i = 1\right) ds$$

$$= -\sum_i p_i \, E\left(\frac{1}{X_i} \,\Big|\, Z_i = 1\right)$$

$$\leq -\sum_i p_i \, E\left(1 - \tfrac{1}{2}(X_i - Z_i) \,\Big|\, Z_i = 1\right)$$

$$\frac{1}{X_i} = \frac{1}{(X_i - Z_i) + Z_i} = \frac{1}{\underbrace{(X_i - Z_i)}_{\text{integer}} + 1} \geq 1 - \tfrac{1}{2}(X_i - Z_i)$$

$$\leq -\sum_i p_i \, E\left(1 - \tfrac{1}{2}(X_i - Z_i) \,\middle|\, Z_i = 1\right)$$

$$= -\mu + \Delta.$$

$\square$