Note Title

1/2/2006

RANDOM GRAPHS

Basic Models

$$G_{n,m} = ([n], E_{n,m})$$

Vertex set $[n] = \{1, 2, \dots, n\}$

Each graph with m edges has same probability

$$\frac{1}{\binom{N}{m}}, \qquad N = \binom{n}{2}$$

$G_{n,p} =: ([n], E_{n,p})$

Vertex set $[n]$

$$\Pr(G_{n,p} = G) = p^{|E(G)|} (1-p)^{N-|E(G)|}$$

i.e. each edge occurs independently with probability p.

Graph property $\mathcal{P}$.

$p = \frac{m}{N}$

$$\Pr(G_{n,p} \in \mathcal{P}) = \sum_{\mu=0}^{N} \Pr(G_{n,p} \in \mathcal{P} \mid |E_{n,p}| = \mu) \times \Pr(|E_{n,p}| = \mu)$$

$$= \sum_{\mu=0}^{N} \Pr(G_{n,\mu} \in \mathcal{P}) \Pr(|E_{n,p}| = \mu)$$

$$\geq \Pr(G_{n,m} \in \mathcal{P}) \Pr(|E_{n,p}| = m)$$

$$\Pr\left(|E_{n,p}| = m\right) = \binom{N}{m} p^m (1-p)^{N-m}$$

$$= (1+o(1)) \frac{N^N \sqrt{2\pi N}\; p^m (1-p)^{N-m}}{m^m (N-m)^{N-m}\, 2\pi \sqrt{m(N-m)}}$$

$m \to \infty$

$N - m \to \infty$

$$= (1+o(1)) \sqrt{\frac{N}{2\pi m(N-m)}}$$

$$\geq \frac{1}{10\sqrt{m}}$$

So,

$$\Pr(G_{n,m} \in \mathcal{P}) \leq 10 m^{1/2} \Pr(G_{n,p} \in \mathcal{P}).$$

Monotone Properties

A property is monotone increasing $\nearrow$ if

$$G \in \mathcal{P} \Rightarrow G + e \in \mathcal{P}$$

e.g. connectivity

Monotone decreasing $\searrow$ if

$$G \in \mathcal{P} \Rightarrow G - e \in \mathcal{P}$$

e.g. planarity

Suppose $\mathcal{P}$ is $\nearrow$. $\qquad p = \frac{m}{N}$

$$\Pr(G_{n,p} \in \mathcal{P}) = \sum_{\mu=0}^{N} \Pr(G_{n,\mu} \in \mathcal{P}) \Pr(|E_{n,p}| = \mu)$$

$$\geq \Pr(G_{n,m} \in \mathcal{P}) \underbrace{\sum_{\mu=m}^{N} \Pr(|E_{n,p}| = \mu)}_{\geq \frac{1}{2} - o(1)}.$$

Central Limit Theorem $\Rightarrow$

$$\Pr(G_{n,m} \in \mathcal{P}) \leq 3 \Pr(G_{n,p} \in \mathcal{P}).$$

Graph Process:

$G_0 = ([n], \emptyset),\ G_1, G_2, \ldots, G_m, \ldots G_N = K_n$

$G_{m+1} = G_m$ plus random edge

G_m and $G_{n,m}$ have same distribution.

Markov Inequality

$X \geq 0$ is a random variable with finite mean μ.

$$\Pr(X \geq t) \leq \frac{\mu}{t}$$

Proof

$$E(X) = E(X \mid X < t)\Pr(X < t) + E(X \mid X \geq t)\Pr(X \geq t)$$

$$\geq t \Pr(X \geq t).$$

$\square$

Chebyshev Inequality

X is a random variable with finite mean μ and variance σ^2.

$$\Pr(|X-\mu| \geq t) \leq \frac{\sigma^2}{t^2}$$

Proof

$$\Pr(|X-\mu| \geq t\sigma) = \Pr((X-\mu)^2 \geq t^2)$$

$$\leq \frac{E(|X-\mu|^2)}{t^2}$$

$$= \sigma^2/t^2.$$

$\square$

First Moment Method

Let X be a random variable with finite mean taking values in $\{0, 1, 2, \ldots\}$.

$$\Pr(X \neq 0) \leq E(X)$$

Proof

$$\Pr(X \neq 0) = \Pr(X \geq 1) \leq \frac{E(X)}{1}.$$

$\square$

Second Moment Method

Let X be a non-negative random variable with finite mean and variance. Then

$$\Pr(X > 0) \geq \frac{E(X)^2}{E(X^2)}$$

Proof

Let $Y = \begin{cases} 0 & \text{if } X = 0 \\ 1 & \text{if } X > 0 \end{cases}$

So $XY = X$

Cauchy-Schwartz inequality* implies

$$E(XY)^2 \leqslant E(X^2)\,E(Y^2)$$

or

$$E(X)^2 \leqslant E(X^2)\,P(X>0).$$

$\square$

* Consider quadratic $E((X+tY)^2) \geq 0$, as a function of t.

Evolution of a random graph.

We look at how $G_0, G_1, \dots G_m, \dots$ evolves.

$\omega = \omega(n)$ denotes some slowly growing function e.g. $\omega = \log n$.

(1) $m \leq n^{1/2} / \omega$

G_m is a matching whp

whp: with high probability
i.e. with probability $1 - o(1)$
as $n \longrightarrow \infty$.

Let $p = \frac{m}{N}$ and Let X_2 = number of paths of length 2 in $G_{n,p}$.

$$E_p(X_2) = 3\binom{n}{3} p^2$$

$$\leq \frac{n^3 \times n}{N^2 \times \omega^2}$$

$$\to 0.$$

$P_r(G_{n,p}$ contains path of length $2) = o(1)$

monotone property

$\Downarrow$

$P_r(G_{n,m}$ contains a path of length $2) = o(1)$.

(ii) $m = \omega n^{1/2}$, $m = o(n)$.

G_m contains a path of length 2 whp

Let $p = \frac{m}{N}$ and $X_2 = \#$ paths length 2.

$$E(X_2) = 3\binom{n}{3} p^2$$
$$\approx 2\omega^2$$
$$\to \infty$$

Does not imply $X_2 \neq 0$ whp.

Let $\mathcal{P}_2$ be the set of all paths of length two in K_n.

Let $\hat{X}_2$ = # of isolated paths of length 2 in $G_{n,p}$

$$\hat{X}_2 = \sum_{P \in \mathcal{P}_2} \mathbb{1}_{P \overset{i}{\subseteq} G_{n,p}}.$$

$$E(\hat{X}_2) = 3\binom{n}{3} p^2 (1-p)^{3(n-3)}$$

$$\geq (1-o(1)) \frac{n^3}{2} \cdot \frac{4\omega^2 n}{n^4} \cdot (1 - 6np)$$

$m = o(n)$
$\Downarrow$
$np = o(1)$

$$\longrightarrow \infty.$$

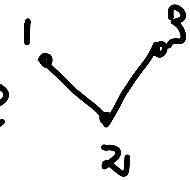
$$\hat{X}_2^2 = \sum_{P \in \mathcal{P}_2} \sum_{Q \in \mathcal{P}_2} \mathbb{1}_{P \overset{i}{\subseteq} G_{n,p}} \mathbb{1}_{Q \overset{i}{\subseteq} G_{n,p}}$$

$$= \sum_{\substack{P,Q \in \mathcal{P}_2 \\ P=Q \text{ or } P,Q \text{ vertex disjoint}}} \mathbb{1}_{P \overset{i}{\subseteq} G_{n,p}} \mathbb{1}_{Q \overset{i}{\subseteq} G_{n,p}}$$

$$E(\hat{X}_2^2) = \sum_{P} \left\{ \sum_{Q} \Pr\left(G_{n,p} \overset{i}{\supseteq} Q \,\middle|\, G_{n,p} \overset{i}{\supseteq} P\right) \right\} \times \Pr\left(G_{n,p} \overset{i}{\supseteq} P\right)$$

Expression inside {} is same for all P.

$$= E(\hat{X}_2)\left(1 + \sum_{Q \cap \{1,2,3\} = \emptyset} \Pr\left(G_{n,p} \overset{i}{\supseteq} Q \,\middle|\, G_{n,p} \overset{i}{\supseteq}\right.\right.$$



$$\left.\left.\right)\right)$$

$$\leq E(\hat{X}_2)\left(1 + \binom{n}{3} p^2 (1-p)^{3(n-6)^{*}}\right)$$

$$\leq E(\hat{X}_2)\left(1 + (1-p)^{-3} E(\hat{X}_2)\right)$$

* conditioning means no edge to {1, 2, 3}

So

$$\Pr(\hat{X}_2 \neq 0) \geq \frac{E(\hat{X}_2)^2}{E(\hat{X}_2)\left(1 + (1-p)^{-3} E(\hat{X}_2)\right)}$$

$$= \frac{1}{(1-p)^{-3} + E(\hat{X}_2)^{-1}}$$

$$\to 1.$$

Thus

$$\Pr(G_{n,p} \supseteq \text{isolated 2-path}) \to 1$$

not monotone

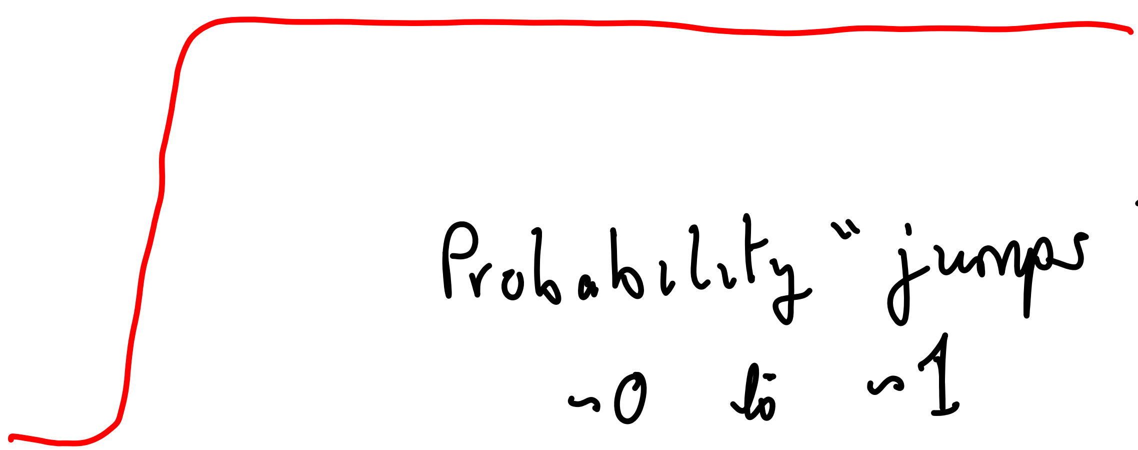
$$\Pr(G_{n,p} \supseteq \text{2-path}) \to 1$$

monotone

$$\Pr(G_m \supseteq \text{2-path}) \to 1$$

$$\Pr\left(G_m \supseteq \text{2-path}\right) = \begin{cases} o(1) & m \ll n^{1/2} \\ 1 - o(1) & m \gg n^{1/2} \end{cases}$$

We say that $n^{1/2}$ is the threshold for the existence of a 2-path in $G_{n,m}$



Probability "jumps" from ~ 0 to ~ 1