Q1 In an inventory system:

- $A$ is the fixed cost associated with making an order.
- $I$ is the inventory charge per unit per period.
- $\pi$ is the back order cost per unit per period.
- $\lambda$ is the demand per period.
- $\psi > \lambda$ is the rate at which ordered items arrive.

Determine the optimal order strategy and its cost per period.

**Solution:**
With reference to the diagram above, we have

\[ Q = \lambda T \]
\[ T = T_1 + T_2 + T_3 + T_4 \]
\[ \frac{T_1 + T_2 + T_4}{T} = \frac{Q - S}{Q}. \]
\[ T_3 = \frac{S}{\lambda T} = \frac{S}{Q}. \]
\[ S = (\psi - \lambda)T_4. \]
\[ h = (\psi - \lambda)T_1. \]
\[ h = \lambda T_2. \]

We deduce from this that
\[ \frac{T_1 + T_2}{T} = \frac{Q - S}{Q} - \frac{T_4}{T} = \frac{Q - S}{Q} - \frac{\lambda}{\psi - \lambda} \cdot \frac{S}{Q} = 1 - \frac{\psi}{\psi - \lambda} \cdot \frac{S}{Q} \]
and
\[ \frac{T_3 + T_4}{T} = \frac{\psi}{\psi - \lambda} \cdot \frac{S}{Q} \]
and
\[ T_1 = (T_1 + T_2) \cdot \frac{\lambda}{\psi} = \left(1 - \frac{\psi}{\psi - \lambda} \cdot \frac{S}{Q}\right) \cdot \frac{Q}{\psi} = \frac{Q}{\psi} - \frac{S}{\psi - \lambda} \]
and then
\[ h = Q \cdot \frac{\psi - \lambda}{\psi} - S. \]

The total cost per period is
\[ K = \frac{A}{T} + I \cdot \frac{T_1 + T_2}{T} \cdot \frac{h}{2} + \pi \cdot \frac{T_3 + T_4}{T} \cdot \frac{S}{2}. \]

This is good enough. No need to do any more for credit.

We write this in terms of \( Q, S \) only.
\[ \frac{A}{T} = \frac{A\lambda}{Q}. \]
\[ I \cdot \frac{T_1 + T_2}{T} \cdot \frac{h}{2} = I \cdot \left(1 - \frac{\psi}{\psi - \lambda} \cdot \frac{S}{Q}\right) \cdot \left(\frac{Q}{2} \cdot \frac{\psi - \lambda}{\psi} - \frac{S}{2}\right). \]
\[ \pi \cdot \frac{T_3 + T_4}{T} \cdot \frac{S}{2} = \pi \cdot \left(\frac{\psi}{\psi - \lambda} \cdot \frac{S}{Q}\right) \cdot \frac{S}{2}. \]
We then have
\[
\frac{\partial K}{\partial Q} = -A\lambda + \frac{I}{2} \left( \frac{\psi - \lambda}{\psi} - \frac{\psi S^2}{(\psi - \lambda)Q^2} \right) - \pi \cdot \frac{\psi}{\psi - \lambda} \cdot \frac{S^2}{2Q^2} \cdot \frac{\psi - \lambda}{Q}.
\]
\[
\frac{\partial K}{\partial S} = I \cdot \left( \frac{\psi}{\psi - \lambda} \cdot \frac{S}{Q} - 1 \right) + \pi \cdot \frac{\psi}{\psi - \lambda} \cdot \frac{S}{Q}.
\]
Putting the partial derivatives equal to zero and solving gives us the optimal values
\[
Q^* = \left( 2A\lambda \cdot \frac{\psi}{\psi - \lambda} \cdot \frac{I + \pi}{I^\pi} \right)^{1/2}.
\]
\[
S^* = \left( 2A\lambda \cdot \frac{\psi}{\psi} \cdot \frac{I}{\pi(I + \pi)} \right)^{1/2}.
\]
\[
K^* = \left( 2A\lambda \cdot \frac{\psi - \lambda}{\psi} \cdot \frac{I\pi}{I + \pi} \right).
\]

**Q2** Analyse the following inventory system and derive a strategy for minimising the total cost. There are \(n\) products. Product \(i\) has demand \(\lambda_i\) per period and no stock-outs are allowed. The cost of making an order for \(Q\) units of a mixture of products is \(AQ^\alpha\) where \(0 < \alpha < 1\). The inventory cost is \(I\) times \(\max\{L_1, L_2, \ldots, L_n\}\) per period where \(L_i\) is the average inventory level of product \(i\) in that period.

**Solution:** We argued in class that we order items at intervals \(T\). Thus we order \(Q_t = T\lambda_i\) units of item \(i\) each time and \(L_i = Q_t/2\). Let \(\lambda = \max\{\lambda_1, \lambda_2, \ldots, \lambda_n\}\). Then the inventory cost is therefore \(IT\lambda/2\).

This gives us a total cost of
\[
K = \frac{A \left( T \sum_{j=1}^{n} \lambda_j \right)^\alpha}{T} + \frac{IT\lambda}{2} = \frac{B}{T^{1-\alpha}} + \frac{IT\lambda}{2}
\]
where \(B = A \left( \sum_{j=1}^{n} \lambda_j \right)^\alpha\).

Now
\[
\frac{dK}{dT} = -\frac{B(1 - \alpha)}{T^{2-\alpha}} + \frac{I\lambda}{2}.
\]

Therefore, the optimal value for \(T\) is given by
\[
T^* = \left( \frac{2B(1 - \alpha)}{I\lambda} \right)^{1/(2-\alpha)}.
\]
Q3  Give an algorithm to solve the following scheduling problem. There are
$n$ jobs labelled 1, 2, . . . , $n$ that have to be processed one at a time on
a single machine. There is an acyclic digraph $D = (V, A)$ such that if
$(i, j) \in A$ then job $j$ cannot be started until job $i$ has been completed.
The problem is to minimise $\max_j f_j(C_j)$ where for all $j$, $f_j$ is a monotone
increasing. As usual, $C_j$ is the completion time of job $j$. This is distinct
from its processing time $p_j$.

Solution: Let $S$ be the set of jobs with no successor in $D$ i.e. the set
of sinks of $D$. The last job must be in $S$ and it will complete at time
$p = p_1 + p_2 + \cdots + p_n$. Let $f_k(p) = \min_{j \in S} f_j(p)$. We schedule $k$ last and
then inductively schedule the remaining jobs.