Q1 Can the following shortest path problem be solved by the Dijkstra algorithm? The edges of a digraph are colored Red, Blue and Green. Suppose edge lengths are non-negative, but a path can have at most \( r \) Red edges, \( b \) Blue edges and no Blue edge can be followed by a Green edge. Give an explicit definition of path length.

**Solution** Let \( \mathcal{R} \) denote the set of restrictions imposed by the colorings. Then the length of a path is given by

\[
\ell(P) = \begin{cases} 
\sum_{e \in P} \ell(e) & P \text{ satisfies } \mathcal{R}, \\
\infty & \text{Otherwise}.
\end{cases}
\]

Clearly \( \ell(P) \geq \ell(Q) \) whenever \( Q \) is a subpath of \( P \) and so we can use Dijkstra’s algorithm.

Q2 Convert the following into a standard assignment problem. We have a bipartite graph with bipartition \( A = \{a_1, a_2, \ldots, a_m\}, B = \{b_1, b_2, \ldots, b_n\} \). An assignment now is a set of edges \( M \) such (i) \( a_i \) is incident to exactly \( r_i \) edges of \( M \) for \( i = 1, 2, \ldots, m \) and (ii) \( b_j \) is incident to exactly \( s_j \) edges of \( M \) for \( j = 1, 2, \ldots, n \). Here \( \sum_i r_i = \sum_j s_j \). The cost of edge \( (a_i, b_j) \) is \( c(i, j) \) and the cost of an assignment \( M \) is \( \sum_{e \in M} c(e) \). The objective is to find a minimum cost assignment.

**Solution** We replace the vertex \( a_i \) by vertices \( a_i(1), a_i(2), \ldots, a_i(r_i) \) for \( i = 1, 2, \ldots, m \) and the vertex \( b_j \) by vertices \( b_j(1), b_j(2), \ldots, b_j(s_j) \) for \( j = 1, 2, \ldots, n \). The cost of edge \( \{a_i(k), b_j(l)\} \) will be \( c(a_i, b_j) \). Each solution \( x \) to the original problem can be mapped to \( \prod_{i=1}^m \prod_{j=1}^n r_i!s_j! \) solutions of the expanded problem, and each of these has the same cost. Each solution to the expanded problem arises from a unique solution to the original problem.
Q3 Let $G = (A, B, E)$ be a bipartite graph. Let $I \subseteq B$ be independent if $G$ contains a matching $M$ that is incident with every vertex in $I$. Show that the independent sets form a matroid.

Hint: consider the action of augmenting paths.

**Solution** Clearly the independent sets form an independence system. Next, suppose that $I_1, I_2$ are independent and that $M_1, M_2$ are matchings incident with $I_1, I_2$ respectively and that $|I_1| = |M_1| > |M_2| = |I_2|$. Consider $M_1 \oplus M_2$. It consists of alternating paths and cycles, but because $|M_1| > |M_2|$ there must be at least one alternating path $P$ that goes from a vertex $a \in A$ to a vertex $b \in B$ such that neither $a$ nor $b$ are covered by $M_2$. Here $b \in I_1 \setminus I_2$. If we amend $M_2$ by removing $M_2 \cap P$ and adding $M_1 \cap P$ then we will obtain a new matching that covers $I_2 \cup \{b\}$. This verifies the axioms of a matroid.