Q1 In an inventory system:

- $A$ is the fixed cost associated with making an order.
- $I$ is the inventory charge per unit per period.
- $\pi$ is the back order cost per unit per period.
- $\lambda$ is the demand per period.
- $\psi > \lambda$ is the rate at which ordered items arrive.

Determine the optimal order strategy and its cost per period.

Solution:
With reference to the diagram above, we have

\[ Q = \lambda T \]
\[ T = T_1 + T_2 + T_3 + T_4 \]
\[ \frac{T_1 + T_2 + T_4}{T} = \frac{Q - S}{Q} \]
\[ \frac{T_3}{T} = \frac{S}{\lambda T} = \frac{S}{Q} \]
\[ S = (\psi - \lambda)T_4. \]
\[ h = (\psi - \lambda)T_1. \]
\[ h = \lambda T_2. \]

We deduce from this that

\[ \frac{T_1 + T_2}{T} = \frac{Q - S}{Q} - \frac{T_4}{T} = \frac{Q - S}{Q} - \frac{\lambda}{\psi - \lambda} \cdot \frac{S}{Q} = 1 - \frac{\psi}{\psi - \lambda} \cdot \frac{S}{Q} \]

and

\[ \frac{T_3 + T_4}{T} = \frac{\psi}{\psi - \lambda} \cdot \frac{S}{Q} \]

and

\[ T_1 = \frac{T_1 + T_2}{T} \cdot \frac{\lambda}{\psi} = \frac{Q - S}{Q} \cdot \frac{\psi}{\psi - \lambda} \cdot \frac{S}{Q} \]

and then

\[ h = Q \cdot \frac{\psi - \lambda}{\psi} - S. \]

The total cost per period is

\[ K = \frac{A}{T} + I \cdot \frac{T_1 + T_2}{T} \cdot \frac{h}{2} + \pi \cdot \frac{T_3 + T_4}{T} \cdot \frac{S}{2}. \]

We write this in terms of \( Q, S \) only.

\[ \frac{A}{T} = \frac{A \lambda}{Q}. \]
\[ I \cdot \frac{T_1 + T_2}{T} \cdot \frac{h}{2} = I \cdot \left( 1 - \frac{\psi}{\psi - \lambda} \cdot \frac{S}{Q} \right) \cdot \left( \frac{Q}{2} \cdot \frac{\psi - \lambda}{\psi} - \frac{S}{2} \right). \]
\[ \pi \cdot \frac{T_3 + T_4}{T} \cdot \frac{S}{2} = \pi \cdot \left( \frac{\psi}{\psi - \lambda} \cdot \frac{S}{Q} \right) \cdot \frac{S}{2}. \]
We then have

\[ \frac{\partial K}{\partial Q} = -A\lambda \frac{Q}{2} + I \left( \frac{\psi - \lambda}{\psi} - \frac{\psi S^2}{(\psi - \lambda)Q^2} \right) - \pi \cdot \frac{\psi}{\psi - \lambda} \cdot \frac{S^2}{2Q^2} \]

\[ \frac{\partial K}{\partial S} = I \cdot \left( \frac{\psi}{\psi - \lambda} \cdot \frac{S}{Q} - 1 \right) + \pi \cdot \frac{\psi}{\psi - \lambda} \cdot \frac{S}{Q} \]

Putting the partial derivatives equal to zero and solving gives us the optimal values

\[ Q^* = \left( 2A\lambda \cdot \frac{\psi}{\psi - \lambda} \cdot \frac{I + \pi}{I\pi} \right)^{1/2} \]

\[ S^* = \left( 2A\lambda \cdot \frac{\psi - \lambda}{\psi} \cdot \frac{I}{\pi(I + \pi)} \right)^{1/2} \]

\[ K^* = \left( 2A\lambda \cdot \frac{\psi - \lambda}{\psi} \cdot \frac{I\pi}{I + \pi} \right) \]

**Q2** Analyse the following inventory system and derive a strategy for minimising the total cost. There are \( n \) products. Product \( i \) has demand \( \lambda_i \) per period and no stock-outs are allowed. The cost of making an order for \( Q \) units of a mixture of products is \( AQ^\alpha \) where \( 0 < \alpha < 1 \). The inventory cost is \( I \) times \( \max\{L_1, L_2, \ldots, L_n\} \) per period where \( L_i \) is the average inventory level of product \( i \) in that period.

**Solution:** We argued in class that we order items at intervals \( T \). Thus we order \( Q_i = T\lambda_i \) units of item \( i \) each time and \( L_i = Q_i/2 \). Let \( \lambda = \max\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \). Then the inventory cost is therefore \( IT\lambda/2 \). This gives us a total cost of

\[ K = A \left( \frac{T \sum_{j=1}^n \lambda_i}{T} \right)^\alpha + \frac{IT\lambda}{2} = \frac{B}{T^{1-\alpha}} + \frac{IT\lambda}{2} \]

where \( B = A \left( \sum_{j=1}^n \lambda_i \right)^\alpha \).

Now

\[ \frac{dK}{dT} = -\frac{B(1 - \alpha)}{T^{2-\alpha}} + \frac{I\lambda}{2} \]

Therefore, the optimal value for \( T \) is given by

\[ T^* = \left( \frac{2B(1 - \alpha)}{I\lambda} \right)^{1/(2-\alpha)} \]
Q3 Give an algorithm to solve the following scheduling problem. There are
$n$ jobs labelled $1, 2, \ldots, n$ that have to be processed one at a time on
a single machine. There is an acyclic digraph $D = (V, A)$ such that if
$(i, j) \in A$ then job $j$ cannot be started until job $i$ has been completed.
The problem is to minimise $\max_j f_j(C_j)$ where for all $j$, $f_j$ is a monotone
increasing. As usual, $C_j$ is the completion time of job $j$. This is distinct
from its processing time $p_j$.

**Solution:** Let $S$ be the set of jobs with no successor in $D$ i.e. the set
of sinks of $D$. The last job must be in $S$ and it will complete at time
$p = p_1 + p_2 + \cdots + p_n$. Let $f_k(p) = \min_{j \in S} f_j(p)$. We schedule $k$ last and
then inductively schedule the remaining jobs.