Optimization

\[ \text{Minimize } f(x) \]
subject to \[ x \in S \subseteq \mathbb{R}^n \]

Linear Programming: \[ f(x) = 5x \] and \[ S = \{ x \mid Ax = b, x \geq 0 \} \]

\( x^* \) is a global minimum if \( f(x) \geq f(x^*) \) for all \( x \in S \).

\( x^* \) is a local minimum if \( f(x^*) \leq f(x) \) for all \( x \) in a neighborhood of \( x^* \).

Let \( S = \{ x \mid x \geq 0 \} \), such that \( f(x^*) \leq f(x) \) for all \( x \geq 0 \).
Convex Function

\[ f : \mathbb{R}^2 \to \mathbb{R} \text{ is convex if } \]

\[ f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y), \text{ for all } \lambda. \]

Examples:
- \( f(x) = e^x \)
- \( f(x) = -\log x \)

Admissible \( \frac{d}{dx} \) is convex.
\[ f(x) = f(y) \leq f(x) + (y-x)f'(x) + \frac{1}{2} (y-x)^2 \]

\[ f(y) \geq f(x) + \nabla f(x) \cdot (y-x) \]

Consider \( h(t) = f(tx + (1-t)y) \), Convex function of \( t \).
If \( f'' \) exists, then \( f \) is convex iff \( f''(x) > 0 \) for all \( x \).

Quadratic Functions

\[ Q(x) = x^T A x = \sum_{i,j} A_{ij} x_i x_j \]

\( A \) is a real matrix

We can assume \( A \) is symmetric:

\[ A = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix} \]

\[ A_{i,j} = A_{j,i} \]

A is positive semi-definite if \( Q(x) \geq 0 \)

If \( Q \) is convex, then \[ Q(x) \geq 0 \] for all \( x \).
Operations on Convex Functions:

1. If $g$ convex $\Rightarrow f + g$ is convex
2. If $f$ convex $\Rightarrow \lambda f$ is convex
3. If $g$ convex $\Rightarrow \max f + g$ is convex

Jensen's Inequality:

\[
\sum_{i=1}^{n} \lambda_i x_i \geq \sum_{i=1}^{n} \lambda_i f(x_i) \quad \text{for $\lambda_i \geq 0$, $\sum \lambda_i = 1$}
\]

\[
\text{Induction on } M, \quad M=2, \text{ base case}
\]
Convex Sets

Non-convex

Convex: $S$ is convex iff for line segment $L(x,y), x,y \in S \Rightarrow L(xy) \subseteq S.$

Examples:

- $S = \{x : x^2 < 1\}$
- $S = \{x : x^2 < 1\}$
- $S = B(x, r) \cap \mathbb{C}$

Level set of $f$:

- $f(x, y) \leq C$ when $B(x, r) \cap \mathbb{C}$
- $f(x, y) \leq C$ when $B(x, r) \cap \mathbb{C}$
Operations on Convex Sets

1. \( S + T \) is convex.

2. \( S \times T \) is convex.

If \( S \) and \( T \) are convex and \( x^* \) is a local minimum of \( f \), then \( x^* \) is also a global minimum.