Department of Mathematical Sciences
CARNEGIE MELLON UNIVERSITY

OPERATIONS RESEARCH II 21-393

Homework 1: Due Monday September 15.

Q1 Suppose that in the simple production problem, we do not have to meet demand immediately. Instead there is a penalty of b dollars per unit per period of demand not met. Suppose that we are only allowed to have at most B units of demand unfulfilled at any stage.
Solution:
We let negative inventory denote demand that is as yet unmet. In which case we have

$$f(r,i) = \min_x \{c(x) + b\max\{0,-i\} + f(r+1, x+i-d_r)\}.$$

The constraints on x are

$$\begin{aligned} x &\geq 0. \\ x+i-d_r &\geq -B. \\ x+i-d_r &\leq H. \end{aligned}$$

Q2: An $m \times n$ rectangle of wood is to be cut into smaller rectangles. An $a \times b$ rectangle is worth $m_{a,b}$. The machine that cuts rectangles can only cut full length or full width. I.e. if after cutting there is an $x \times y$ rectangle then the machine can cut it into two rectangles $z \times y$ and $(x-z) \times y$ for some z or into two rectangles $x \times z$ and $x \times y - z$. All rectangles cut must have integral side lengths.
Describe a dynamic programming algorithm for finding the way of cutting into pieces that maximises the total value of the rectangles produced.
Solution: Let $f(i,j)$ be the maximum value obtained from a rectangle with corners $(0,0)$ and (i,j). Then

$$f(i,j) = \max \begin{cases} \max_{x \leq i}(m(i-x,j) + f(x,j) \\ \max_{y \leq j} m(i,j-y) + f(i,y) \end{cases}$$

Q3 Consider a 2-D map with a horizontal river passing through its center. There are n cities on the southern bank with x-coordinates $a(1)...a(n)$ and

n cities on the northern bank with x-coordinates $b(1)...b(n)$. You want to connect as many north-south pairs of cities as possible with bridges such that no two bridges cross. When connecting cities, you can only connect city i on the northern bank to city i on the southern bank. Construct a Dynamic Programming solution to this problem. (You can assume that $a(1) < a(2) < \cdots < a(n)$, but you **cannot** assume that $b(1) < b(2) < \cdots < b(n)$. If both sequences are increasing, then the problem is trivial).

Solution: Let $f(j)$ be the maximum number of bridges choosable if we only use $(a(i), b(i),\ i \geq j)$. Then

$$f(j) = \max \begin{cases} f(j+1) & \text{do not choose } (a(j), b(j)) \\ 1 + f(\min\{k > j :\ b(k) > b(j)\}) & \text{choose } (a(j), b(j)) \end{cases}.$$

Q4 You have to drive across country along a road of length L. There are gas stations at points $p_1, p_2, \ldots, p_r$ along the route. Your car can hold g gallons of gasoline. At gas station i, the price of gas is p_i per gallon. If you drive at s miles per hour then you use up $f(s)$ gallons of gas per mile. You can assume that you have to drive at constant speed between stops. You start with a full tank of gas and you have an amount A to spend on the trip. Can you finish the trip in time at most T?

Solution: Let $\theta(i, t, \gamma, a) = 1$ if it is possible to finish the trip within time t starting at p_i with a to spend and γ gallons of gas and let $\theta(i, t, \gamma, a) = 0$ otherwise. Let $P_0 = 1$. We have to determine whether $\theta(0, T, g, A) = 1$. For this we use the recurrence,

$$\theta(i, t, \gamma, a) = \max_{j>i,s,b} \left\{ \theta \left(j, t - \frac{P_j - P_i}{s}, \gamma + b - (p_j - p_i) f(s), a - bp_i \right) \right\}$$

Here j represents the choice of next stop, s represents the speed you will travel and b denotes the amount of gas that you will buy at station i. The constriants are

$$\begin{aligned} \gamma + b - (P_j - P_i) f(s) &\geq 0. \\ \gamma + b &\leq g. \\ a - bp_i &\geq 0. \\ t - \frac{P_j - P_i}{s} &\geq 0. \end{aligned}$$